Tsukuba Journal of Educational Study in Mathematics. Vol.18, 1999

ARTICLE

Cognitive Structures Underlying Conceptions of Mathematical Proof

Yasuhiro SEKIGUCHI
Faculty of Education, Yamaguchi University

This paper is an exploratory attempt to analyze and describe cognitive structures underlying conceptions of mathematical proof appearing in mathematics instruction. Based on the framework of cognitive semantics, cognitive models, mental space mappings, and narrative structures in conceptions of proof are analyzed. JOURNEY metaphor, SHOWING metaphor, modular schema, organic schema, and two different narrative structures are found. Discussing the results, the paper proposes the use of the narrative structure of adventure story in the instruction of mathematical proof.

KEY WORDS: proof, cognitive model, schema, narrative

This study aims to describe how people conceive mathematical proof: What meanings do they give to it? How do they relate it to other experiences? How do they structurize mathematical proof. To capture people's conceptions of proof, the current study uses the framework of cognitive semantics (espoused by Gilles Fauconnier, Mark Johnson, George Lakoff, and so on). The present paper tries to elucidate cognitive models, mental space mappings, and narrative structures underlying conceptions of mathematical proof in mathematics instruction. Based on the results the paper discusses a new conceptualization of mathematical proof for mathematics instruction.

Metaphors and Schemas Underlying Conceptions of Proof

One of the most basic cognitive processes is categorization: It involves how we perceive things, develop concepts, and act to the world. Lakoff (1987) proposes "experiential realism" as a framework of human categorization. According to this, people conceptualize their experiential world through their own conceptual structures, and construct a reality. Lakoff calls the conceptual structures "idealized cognitive models" (ICMs)--sometimes, simply, "cognitive models." The ICM is a mental construct for organizing experiences. The ICMs are formed based on "preconceptual" experiences. They consist of two types:

A. Basic-level structure: categories formed from gestalt perception, bodily movement, and mental images.

B. Kinesthetic image-schematic structure: categories formed from relatively simple

> structures recurring in everyday bodily experiences. Examples include “containers,” “paths,” “links,” “forces,” “balance,” “up-down,” “front-back,” “part-whole,” “center-periphery” (Lakoff, 1987, p. 267).

The ICMs are constructed through various “mappings” from these preconceptual structures.

According to Johnson (1987), and Fauconnier (1994, 1997), mappings are essential in meaning construction; they work for connecting and generating "mental spaces." A mental space is any mental structure which represents a state of affairs as understood. It is what one conceives as a “situation,” “domain,” “state,” “reality,” or “world,” whether one considers it as real, fictional, past, present, future, concrete, or abstract. Fauconnier (1997) identifies three types of mapping: schema mapping, projection mapping, and pragmatic function mapping. A schema mapping is used when a cognitive model structures a mental space. A projection mapping is to project part of the structure of one mental space to another. One of the most important projection mappings in human understanding is metaphor (or analogy): It connects a familiar domain ("source") to another domain ("target"), which is in question, with guide of a common cognitive model (pp. 102-105). That cognitive model structures both domains by schema mappings. [Following Lakoff's notation system (1993, p. 207), I refer to a metaphor from a source domain X to a target domain Y by “Y IS X.”] Lastly, a pragmatic function mapping is to connect two relevant mental spaces by a “pragmatic function” (defined by G. Numberg). Metonymy is a typical instance of that kind of mapping.

PROOF IS A JOURNEY Metaphor

Proof of a statement is commonly described as a series of justified statements that “starts” from the given information and “ends” at the statement in question: Underlying is a “journey” metaphor (Lakoff & Johnson, 1980). Some school textbooks illustrate this image of proof using a flow chart diagram. This conceptualization is based on the “source-path-goal schema”: It is a kinesthetic image-schematic structure from such everyday experience that “[e]very time we move anywhere there is a place we start from, a place we wind up at, a sequence of contiguous locations connecting the starting and ending points, and a direction” (Lakoff, 1987, p. 275). This consists of “a SOURCE (starting point), a DESTINATION (end point), a PATH (a sequence of contiguous location connecting the source and the destination), and a DIRECTION (toward the destination)” (p. 257). The given information of a problem is a source, the statement to be proved is a destination, and a proof is a path directing from the source to the destination.

Many Japanese textbooks first introduce proof “sho-mei” as showing or telling why something is true along a well-ordered line ("suji-michi wo tatete"). And, later when the idea of conditional statement ("if..., then ...") is discussed, and when the terms "assumption (or hypothesis)" (katei) and "conclusion" (ketsuron) are introduced, proof is defined as leading to the conclusion, *starting from* the assumptions, using those things which are accepted: The assumptions are given the status of the starting point, not just things accepted (Figure 1).

[assumptions]
↓←things having been accepted true
[conclusion]

Figure 1. Mechanism of proof (From Fukumori et al., 1996, p. 106)

Here, the assumptions [katei] are a source, and the conclusion [kesturon] is the destination. Proof is a path from the former to the latter. Reasons are enabling conditions for moves. The person who proves is required to "show" the path of journey.

PROOF IS SHOWING Metaphor

The verb "show" is almost a synonym of "prove" in the mathematical discourse. In fact, proof problems in textbooks very often end the statements by saying "Show that ... ." Proof is conceived as a tour guide of letting tour participants "see" a path connecting the initial situation and the conclusion (cf. Lakoff, 1993, pp. 243-244; Lakoff & Johnson, 1980, p. 103). Mathematicians complain "I can't see" or "It's not clear" when they fail to "follow" the guide, and praise "It's insightful," or "That's illuminating" when they reach a spectacular point of the tour. Here appear two different metaphors: the journey metaphor as discussed above, and the UNDERSTANDING IS SEEING metaphor (Johnson, 1993, p. 224; Lakoff & Johnson, 1980, p. 48). The latter metaphor is a mapping from the domain of vision to the domain of knowing (Johnson, 1993, p. 224):

Visual Domain		*Understanding / Knowing*
Eye (organ of sight)	⟶	Mind's eye (organ of knowing)
Physical object seen	⟶	Ideas (mental object)
Light	⟶	(Light of) reason
Visual field	⟶	Domain of knowledge
Visual acuity	⟶	Mental acuity

Proof is a guide of letting tourists see how the journey goes.

The UNDERSTANDING IS SEEING metaphor has been deeply rooted in western traditional philosophies. In the history of mathematics, according to Platonism, mathematical objects are abstract entities in the Platonic universe; they exist independently of any human activity, and each of them has an unchangeable nature, an "essence." In spite of the nonhuman character of mathematical objects, Platonists say that essences and truths about Platonic entities are accessible to human beings: They can be received into the individual human soul by "intellectual intuition"--*mind's eye*. The proof procedure in Platonism consists in the derivation of a theorem by following valid inferences from "trivial truths" (axioms, or postulates), *perfectly-clear* descriptions of essences (definitions), and previously derived theorems. Also, Descartes's (1701/1954) Rule III contended that to acquire knowledge, "we must inquire . . . what we can clearly and manifestly perceive by intuition or deduce with certainty" (p. 153). For him, proof consisted in deducing "from principles known to be true by a continuous and uninterrupted movement of thought, with *clear intuition of each point*" (p. 156. Italics added).

Modular Schema

In the formalist mathematics, a mathematical statement is represented in terms of mathematical and logical symbols. A proof is represented by a set of atomistic steps guided by rules of inference, and each step consists of a mathematical statement. This formalist construction of mathematics is structured through the modular schema. That schema was identified by anthropologist B. Shore (1996) in many of modern artifacts like furniture, assembly lines, shopping molls, personal computers, school curriculum, and so on. It is “a design strategy that breaks complex wholes into elementary units that are understood to be recombinable into a variety of different patterns” (Shore, 1996, p.118).

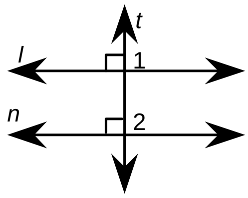
Influence of the formalistic mathematics encouraged some mathematicians reducing the proof procedure to symbolic manipulation, so that an "objective" standard of proof can be achieved. In the United States the two-column form is a common format for writing proofs in high school geometry (Figure 2), and sometimes in algebra also. In this format a proof looks like a spreadsheet calculation. One draws a diagram to illustrate the given information and lists in terms of the diagram the given information and the statement to prove. Then, one draws a long horizontal line and a vertical line downward from the middle to form a letter T, creating two columns under the horizontal line. In the left column, one writes a deductive sequence of statements leading to the statement to prove, numbering each statement. For each step of the deduction one has to write in the right column a reason for the deduction with a corresponding number. Because the format arranges a proof in two columns, it was called the two-column form in the textbook. In almost every case, textbooks write the given information as the statement numbered 1 and "given" as the reason.

The two-column form is structured through the modular schema. In fact, proof is a sequence of steps, a combination of atomistic modules. Any of the components has equal status as a numbered member filling a column of a line. No explicit reference to previous steps is usually indicated. (Indirect proofs are usually written in the paragraph form because writing an indirect proof in the two column form is rather complicated.)

If a transversal is perpendicular to one of two parallel lines, then it is perpendicular to the other one also.

Given: Transversal t cuts l and n;
$t \perp l$; $l \parallel n$

Prove: $t \perp n$



Proof:

Statements	Reasons
1. $t \perp l$	1. Given
2. $m\angle 1=90$	2. Def. of $\perp$ lines and def. of a rt.$\angle$
3. $l \parallel n$	3. Given
4. $m\angle 2 = m\angle 1$	4. If two parallel lines are cut by a transversal, then corr. $\angle$s are =.
5. $m\angle 2 = 90$	5. Substitution Property
6. $t \perp n$	6. Def. of a rt. $\angle$ and def. of $\perp$ lines

Figure 2. A two-column form proof (Jurgensen, Brown, & Jurgensen, 1988, p. 61).

Organic Schema

Proofs in geometry in Japanese junior high schools are usually written in “paragraph form,” the writing style in ordinary life (Figure 3).

<u>In triangle ABC, if $\angle$B = $\angle$C, then AB = AC</u>
(Draw the bisector of $\angle$A. Let its intersection with BC be O.)
[Proof] In $\triangle$ABO and $\triangle$ACO,
From the assumptions, $\angle$B = $\angle$C —①
$\angle$BAO = $\angle$CAO —②
Because the sum of the interior angles of a triangle is 180,° from ① and ②,
$\angle$AOB = $\angle$AOC —③
Because AO is common,
AO = AO —④
Because, from ②, ③, and④, one side and the angles at its both ends in each triangle are equal to the corresponding parts of the other, respectively,
$\triangle$ABO ≡ $\triangle$ACO
Therefore, AB = AC

Figure 3. A proof in Japanese textbook.

Unlike the two-column form, a proof here is structured through the organic schema, which comes from our bodily organ system (cf. Johnson, 1993, p. 55): It is not a combination of equal status elements. A statement may have a different role from the others. Connections are emphasized among the statements.

The format of writing geometric proof are gradually introduced in Japan. It looks much similar to writing solutions for algebraic problems. In the second year of junior high school mathematics, students learn algebraic topics. They learn algebraic problems to explain (e. g. “Explain why the sum of two odd numbers is even”). The solutions are written in ordinary writing form. Then they learn solving systems of linear (simultaneous) equations. In writing a solution they are asked to number each equation and use the numbering to organize the answer (Figure 4):

Solve the following simultaneous equations:

$$\begin{cases} 5x - 2y = 6 \\ y - 2x = 1 \end{cases}$$

Solution: 5x - 2y = 6 .....①
y -2x = 1 .....②
From ②, y = 1 + 2x .....②'
Putting ②' into ①, 5x - 2(1 + 2x) = 6
5x - 2 -4x = 6
x = 8
Putting x = 8 into ②', y = 17
(x, y) = (8, 17)

Figure 4. Simultaneous equations (Fukumori et al., 1996, p. 31)

The above learning shapes the background of the style of proof in geometry. Many of the relationships to prove in geometry are equality or congruence: AB = BC, △ABC ≡△DEF. The appearance is close to algebraic writing: Students can assimilate to previous learning in algebra. Also, unlike the two-column form, students do not have to write a reason for every statement. Main focus of learning is to use geometrical knowledge. Emphasis is on those processes where geometric reasons are used.

Narrative Aspects of Proof

In the previous section the central importance of cognitive models and mappings for making sense of experiences was discussed. These are fundamental cognitive resources for categorizing experiences and constructing concepts. The activity of human understanding consists in organized uses of these resources. There is another type of cognitive resources which plays a crucial role in organizing human experiences. They are “narratives,” which provide contexts--meanings and relevance--for individual models or mappings in use. A narrative originally signifies a story told or written to others in words. Following Johnson (1993), I extend the notion of the narrative to a comprehensive cognitive structure for organizing temporal human experiences. Narratives are cognitive structures that shape a temporal series of fragmentary events into a meaningful, coherent whole:

> What is it that distinguishes disconnected, unrelated, and episodic events falling randomly into sequences from meaningful actions? The answer is the synthetic unity supplied by

> cognitive models, metaphors, frames, and narratives--the overarching ordering that transforms mere sequences of atomic events into significant human actions and projects that have meaning and moral import (Johnson, 1993, p. 165).
>
> My claim is that most of this synthesizing activity is done by imaginative structures and that narrative structure provides the most comprehensive synthetic unity that we can achieve. (p. 170)

One of the prevalent structures of narratives is the SOURCE-PATH-GOAL schema already discussed above. The schema recurs in protagonist' activities in a story, and ways of reading the story (pp. 166-169). Another prevalent structure is BALANCE schema: "A story will often begin with an initial situation characterized by a balanced, harmonious state, ... But this balance is soon upset, creating a deep tension. ... The story develops around attempts to restore the initial harmony or to forge some new harmony that restores the normative BALANCE" (p. 169).

For mathematical proof, narrative activities are essential. A proof involves description, explanation, or justification of a mathematical activity in words. It is a discursive activity giving "reasons"--accounts on how one arrived a conclusion. And, the "tour guide" in the PROOF IS SHOWING metaphor corresponds to "a narrator."

There are several different types of narrative structure in proof. Hanna (1990) pointed out that there were two types of proof: A "proof that proves" and "a proof that explains." This distinction seems to be closely related to a narrative difference. Hanna illustrated those types using proofs of "the sum of the first n positive integers is $n(n + 1)/ 2$." Her example of the former is a proof by mathematical induction:

> For n = 1 the statement is obviously true.
>
> Assume it is true for an arbitrary positive integer k. Then consider the sum $1 + 2 + ... + k + (k + 1)$. By the assumption of induction, the sum equals to $[k (k + 1) / 2] + (k + 1)$. Calculating it, we obtain $(k + 1)(k + 2) / 2$, that is, $(k + 1)[(k + 1) + 1] / 2$. This shows that the statement hold true for $k + 1$.
>
> Therefore, the statement is true for all positive integer n.

Her example of the latter type is the famous proof by Gauss in his childhood (though, in the legend, n was 10).

Gauss's proof :

> We want to find out the sum of the first n integers $1, 2, ... , (n - 1), n$. Consider its reverse sequence $n, n - 1, ... , 2, 1$, and add the ith term of the first sequence and that of the reverse sequence. We then obtain third sequence $1 + n, 2 + (n -1), ... , (n -1) + 2, n + 1$. This sequence is actually n times repetitions of $n + 1$. The sum of that sequence is $n (n + 1)$. This must be

equal to twice the sum of 1, 2, ... , (n - 1), n because it is [1 + 2 + ... + (n - 1) + n] + [n + (n - 1) + ... + 2 + 1], and the reversing the order of addition does not change the value of the sum. Therefore, the sum of the first n integers must be half of n (n + 1).

An important difference between these types consists in the narrative structure of each type. The former uses "a confirmation story of the already known result," and the latter an imaginary "discovery story." In the proof by mathematical induction, the story begins assuming that one already knows the final formulation $n(n+1)/2$. There is no description about the journey of how one found the formulation. Therefore, the proof is just for making it solid. On the other hand, in Gauss's proof, the story does not assume that one knows the final formulation, imaging or pretending as if one does not know it. One obtains the result at the end of the journey. The proof shows the journey of how one arrives at the formulation.

Narratives cannot be separated from affective processes. Proof by induction does not accompany emotional excitements because it is just verification, nothing unexpected happens. But Gauss's would be different. Imagine that you are talkig in your class about a story of a child Gauss and explaining his proof, maybe using drawings of dots or a staircase (see Hanna, 1990, p. 11). When your talk reaches the point "This sequence is actually n times repetitions of n + 1," students may express emotionally distinct reactions like surprises, joys, or excitements.

Proposals for A New Conceptualization of Proof

Proof as Adventure Story

Stories of problem solving processes often follow the narrative structure of an adventure story. In adventure stories, protagonists try to reach a challenging goal. They face many obstacles before reaching there. But employing ingenious means, they find breakthroughs. Finally, they reach the goal, with excitement. A "breakthrough"--finding an insightful idea, or inventing a novel theory--in the problem solving is usually called a "discovery": It is typically an emotional moment, experiencing "Aha!" The pioneer of heuristics, George Polya stressed the importance of "discovery" in mathematical problem solving:

> A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. (Polya, 1957, p. v)

Towards a new conceptualization of mathmatical proof in mathematics instruction, *first I contend that the narrative structure for proof should be also the adventure story.* This is because I believe that whether it is a "problem to prove" or "problem to find" (Polya, 1957), solving a mathematics problem should be considered to form an exciting story.

Second, the use of paragraph form writing of proof should be encouraged because it is much closer to ordinary style of story-telling than the two-column form. In addition, the spreadsheet-like appearance of the two-column form may turn off students' emotional reactions for adventure.

The Starting Point of Proof

As discussed above, many school textbooks describe that a proof "starts" from the hypothesis, givens, or assumptions. If a proof is conceived as an adventure, however, the starting point should be a problem situation like any problem solving activity, not the assumptions. For example, think of, again, the algebraic proof problem "Prove that the sum of the first n positive integers is $n(n + 1)/ 2$." Asking what the assumptions in this problem are (e.g., n is a positive integer) does not seem very helpful. A natural start of inquiry would be to pose a question what $1 + 2 + 3 + 4 + ... + n$ would be, then try to manipulate it, represent it with objects, a table, or a graph, or consider similar problems.

In symbolic logic, conditional statement "if a, then b," where a and b are propositions, is often represented by "$a \rightarrow b$", and may suggest that a is the start of reasoning. But this can be represented by either "$a \supset b$," or "$\neg a \vee b$". The conditional statement is claiming that b holds true in any possible situation where a holds true. No necessity that a must be the "start" of proof. Indeed, in symbolic logic, proof of proposition p is defined as a sequence of propositions p_1, p_2, ..., p_n $(= p)$, where p_i is an axiom, or a theorem, or derived from previous propositions by rules of inference. The p_1 does not have to be an assumption in the p.

Polya (1957) mentioned strategies for problem solving: Asking what are the data, what is the condition. Working from "hypothesis," "the assumptions" or "givens" is mere a strategy of solving proof problems: "Could you derive something useful from the hypothesis?" (Polya, 1957, p. 156). There is no intrinsic reason that it must be the starting point of proof problem. In terms of the journey metaphor, the starting point is a location; the assumptions are enabling conditions. It is too restrictive and misleading to demand that students should write "the assumptions" at the start of proof. I observed in classrooms that many students copied down at the top of proof the statements written at the "assumptions" or "givens" of the problem almost as a ritual, whether they were directly used later or not, and number them, whether they were later referred to or not. No wonder that students cannot appreciate how the assumptions affect the conclusions.

Finding a proof is a genuine mathematical inquiry but not a ritual practice. A "problem to prove" is asking one to find out what conclusions would be obtained, pretending that one does not know the conclusions yet. The problem, thus, can be immediately reframed into a "problem to find." We should reconsider mathematical proof within general framework of mathematical problem solving. Therefore, *third, I contend that the starting point for mathematical proof should be a problem situation*.

Metaphorical Introduction of Proof

Fourth, I propose introducing mathematical proof using a metaphorical mapping from "adventure" story: The source is adventure, the target is proof, and the cognitive model to guide the mapping is JOURNEY schema ("source-path-goal schema"). I propose here extensions of the mapping by introducing two components (*roles*) to the source and target domains (Table 1): TOOLS (available means) and IDEAS (journey plans). In learning proofs, students need to be able to analyze their reasoning processes. I expect that introducing those components would help them do the analysis.

Table 1

Correspondences Between ADVENTURE and PROOF

	ADVENTURE	PROOF (EXPLANATION)
SOURCE	a starting location	a given problem situation
DESTINATION	a goal location	a new theorem
PATH	a course of moving from the start to the goal, filled with a lot of danger and excitement	a process of reasoning
DIRECTION	toward the goal	toward a new theorem
[TOOLS]	physical vehicles, instruments, or devices available	definitions, postulates, already proved theorems, the conditions of the given problem, [rules of inferences]
[IDEAS]	plans, strategies, or tactics to cope with difficult situations	plans, insights, intuition, or strategies to deduce the theorem

The use of this metaphorical mapping allows us to talk about proof using the knowledge of adventures. For example, consider the rule "you can't use the conclusion when doing its proof." It is well-known that students often violate this rule. When they do it, we can warn them, "It's in the goal. You can't use it because you haven't reached the goal."

Externalization of Proof

One of the important roles of mathematical proof is systematization of mathematical propositions: Proofs connect definitions, postulates, and theorems each other by illuminating logical relationships among them. This systematization keeps mathematical adventures from becoming just a collection of trial-and-error efforts. Therefore, *fifth, I propose that students prepare notebooks to organize their own writings of proof.* I believe that those notebooks would help students to "externalize" their own learning, i. e., to create those materialized objects which enable them to *show* and *narrate* their learning to other people, and communicate with them. Bruner (1996) summarizes important potentials of "externalization":

> Externalizing, in a word, rescues cognitive activity from implicitness, making it more public, negotiable, and "solidary." At the same time, it makes it more accessible to subsequent reflection and metacognition. (pp. 24-25)

In Fawcett (1938)'s famous experiment also, students created their own notebooks "A Theory of Space" during the course:

> The teacher discouraged any attempt by the pupils to memorize the definitions and assumptions accepted. On the other hand, each pupil was encouraged to use his text freely and to refer to whatever definitions and assumptions he needed in the development of his work. This served to emphasize the importance of his text and was a strong factor encouraging him to keep it neat, well organized and always up to date. As new definitions and assumptions were made they were written in the text with numerous illustrations and supplementary comments, depending on the interests and abilities of the individual pupils. (p. 45)

I believe that this practice was crucial to the success of Fawcett's experiment. Since no formal textbook was used, without using the notebooks, students would not have been able to keep track of what they discussed and accepted in the class. Also, writing down their thinking would have helped them to reflect on it, so that enhancing their critical thinking, which was the main goal of Fawcett's experiment. Unlike commercial textbooks, the notebooks were created by students themselves. This must have facilitated students' awareness of their responsibility for their learning.

Sixth, it is important that students frequently share their writings with others by presenting them in class. I believe that presenting and discussing in class would facilitate students' understanding of *showing* and *narrative* aspects of mathematical proof. Furthermore, it would facilitate the growth of their thinking and writing. According to Vygotsky's theory, intermental activities facilitate the growth of intramental abilities. I believed that presentation and discussion in small groups or in a whole class would stimulate the internal growth of mathematical reasoning, and that oral explanations to the other people would facilitate development of student's ability of written explanation.

Concluding Remarks

Human cognition is mediated by cognitive structures. Cognitive semantics, though a relatively new discipline, is currently providing powerful tools to analyze and describe them. Understanding cognitive models, mappings, and narrative structures underlying our conceptions will be important for improving our instruction. The present paper is an exploratory attempt to capture cognitive structures related to mathematical proof. I hope that it may help improving the current situation of instruction of proof.

*This study was funded by 1996-1997 Grant-in-Aid for Scientific Research (C) of Ministry of Education, Science, Sports and Culture (Project No. 08837013).

References

Bruner, J. (1996). The culture of education. Cambridge, MA: Harvard University Press.

Descartes, R. (1954). Descartes: Philosophical writing (E. Anscombe & P. T. Geach, Trans.). London: Nelson. (Original work published 1701)

Fauconnier, G. (1994). Mental spaces: Aspects of meaning construction in natural language. New York: Cambridge University Press.

Fauconnier, G. (1997). Meanings in thought and language. New York: Cambridge University Press.

Fawcett, H. P. (1938). The nature of proof (the 13th yearbook). Reston, VA: NCTM. (Reprinted in 1995)

Fukumori, N. et al. (1996). Sintei sugaku (second year). Osaka: Keirinkan.

Hanna, G. (1990). Some pedagogical aspects of proof. Interchange **21**(**1**), 6-13.

Johnson, M. (1987). The body in the mind: The bodily basis of meaning, imagination, and reason. Chicago: University of Chicago Press.

Johnson, M. (1993). Moral imagination: Implications of cognitive science for ethics. Chicago: University of Chicago Press.

Jurgensen, R. C., Brown, R. G. & Jurgensen, J. W. (1988). Geometry. Boston: Houghton Mifflin.

Lakoff, G. & Johnson, M. (1980). Metaphors we live by. Chicago: University of Chicago Press.

Lakoff, G. (1987). Women, fire, and dangerous things. Chicago: University of Chicago Press.

Lakoff, G. (1993). The contemporary theory of metaphor. In A. Ortony (Ed.), Metaphor and thought (2nd ed.) (202-251). Cambridge: Cambridge University Press.

Polya, G. (1957). How to solve it (2nd ed.). Princeton, NJ: Princeton University Press.

Shore, B. (1996). Culture in mind. New York: Oxford University Press.