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CONTRIBUTION

Crucial Issues in Teaching of Symbolic Expressions[1)]

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By symbolic expression I mean alpha-numerical expression, that is, expression composed of numerals and alphabetical letters as well as mathematical signs.

Use of letter standing for an unknown number dated back to Diophantus in the 3rd century. Viète in the 16th century expressed known constants as well as unknown numbers with letters, and Descartes in the 17th century almost completed current system of symbolic expressions (Kieran 1990, 1992; Nakamura 1962). Comparing with the long history of mathematics, use of symbolic expressions can be said relatively new. With the use of symbolic expressions, progress of mathematics in general, algebra and calculus in particular, is remarkable, and their application to physical sciences built up the foundation of modern technology and civilization.

Today, introduction to symbolic expressions is one of the most fundamental content in secondary mathematics education for all students (DES & WO 1989; MOE 1989, 1999; NCTM 1989, 2000). Many teachers and students find difficulties in teaching and learning of symbolic expressions, and many research studies and conferences focus on them (Algebra Working Group to NCTM 1995; Bednarz et al. 1996; Bell 1995; Coxford 1988; Dossey 1998; Kaput 1995, 1998; Kieran 1990, 1992, 1996; Lacampagne 1995; NCTM & MSEB 1998; OSGME 1987; Royal Society & JMC Working Group 1995; Steen 1995; Wagner & Kieran 1989a, 1989b). This Regular Lecture focuses on this problem. Considering limit of time, I will concentrate on the foundation of the subject and teaching at introductory stage.

My lecture consists of the following two sections and Concluding Remarks:

1 Nature of Symbolic Expression and Its Use

This is the foundation of the subject, on which I proceed discussion and analysis in the next section. Firstly characteristics of symbolic expression as mathematical language are discussed, and then its wide use and where it is effective are given with a few examples.

2 Teaching of Symbolic Expressions at Introductory Stage

Firstly my proposal for teaching of symbolic expressions at introductory stage is presented. Then three processes of language aspect, which form obviously core part of teaching of symbolic expressions, and the relation to teaching of arithmetic in elementary school as basis for the teaching are considered. Finally technology and teacher, which are most influential factors to teaching, are discussed.

In preparation for this lecture, I owe much to many scholars and researchers in mathematics education around the world through their research papers, articles, documents and books on the subject. They are too many to list their names here. At the start, I would like to express my heartfelt thanks to all these scholars and researchers.

1 Nature of Symbolic Expression and Its Use

(1) Nature and characteristics of symbolic expression

A. Symbolic expression as a mathematical language

In mathematics, symbolic expression is most important and indispensable means of communication and thinking, and it is regarded as mathematical language. It has the following characteristics (MOE 1989, 1999; Sfard 1987, 1991):

(a) It is clear and concise.

(b) It can express the general.

(c) It can be transformed formally.

(d) It expresses process and also is manipulated as an object.

As to (a), symbolic expression is thorough and of no redundancy. It contains much content and meaning. This can be seen obviously in comparison with everyday language. It is because that only the aspect of quantities and their relations are focused on and expressed while others are ignored. For instance, in solving a word problem with use of equation, which is a typical example of symbol expression, an equation is made up by ignoring various nouns, verbs, adjectives and adverbs and so on in the problem but focusing on a certain quantitative relation, and solution is obtained by solving the equation. Further, symbolic expressions represent result which are based on underlying very many concepts and principles. Like an example $E = mc^2$, we can find many symbolic expressions containing profound content in a brief form.

As to (b), while number expression is confined to a particular value of quantity, symbolic expression can express the general. In fact, with use of symbolic expression we represent the general beyond the individual and the particular. Therefore it has ability to elucidate mechanism and structure working in a situation.

The characteristics (c) is vital. We can say that it is the reason why symbolic expression is considered to be a means of thinking in mathematics. Sometimes transformation itself can work in place of thinking. Moreover, what is important is that transformation can be done formally, that is, without taking meaning of letters and context of symbolic expressions into consideration.

Characteristic (d) is different slightly from the above three but must be noticed. Just like formula of quadratic equation denotes the procedure to ask for solutions originally, symbolic expression represents a process. It allows us making aware of the process by writing out in a form of expressions. At the same time, symbolic expression is manipulated as an object, like sum of solutions of a quadratic equation in formula is computed. Understanding the process-product duality of symbolic expression is important.

B. Language aspect of symbolic expression

From the viewpoint of language, the relation between "signified" and "signifier, symbolic expression" is considered. Transition process from signified to signifier is process "to express" and the inverse process is "to read". Transition process between symbolic expressions is "to transform" (Miwa 1996, 1998, 1999). Roles of the three processes in the use of symbol expressions are sketched in a triangular diagram below, which I call Scheme of Use of Symbolic Expressions (Miwa 1996, 1998, 1999).

Starting from a situation and going through a cycle of the three processes, it can be expected that would make discoveries and gain insight in the situation. Therefore, the starting situation may be also assumed to be a terminal.

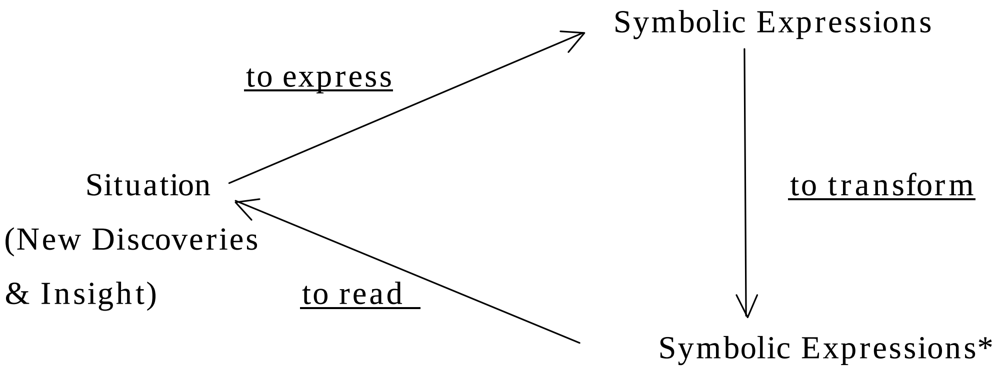


Figure.1 Scheme of Use of Symbolic Expressions

C. User of mathematical language

As to language user, it should be pointed out that illiteracy comes out when education of language is not done adequately. The illiteracy in symbolic expression is serious because it leads to non-use of mathematics and shut off the gate to academic and professional career in future. Foreseeing technology society based on mathematics, an urgent task for us mathematics educators is to prevent the illiteracy of this kind (Dossey 1998; Kaput 1995, 1998; Royal Society & JMC Working Group 1995).

In addition, it should be noted that language is deeply rooted in culture and that its user is influenced largely by their cultural background. For instance, students' difficulties in the well-known Student-Professor Problem is considered to be caused by representation of ratio in English and translation to symbolic expressions (Herscovics 1989; Lockhead & Mestre 1988).

Student-Professor Problem is a question "Write an equation using variables S and P to represent the following statement: There are six times as many students as professors at this university. Use S for the number of students and P for the numbers of professors." The rate of correct responses of 150 freshman engineering students at a major state university in the US was 63%, and 68% of errors were reversals : $6S = P$.

Interpretation of "letter as object" (Küchemann 1981), that is, the one that $2a+5b+a$ was interpreted as 2 apples and 5 bananas and another apple by some UK students is another example of this influence. This interpretation may be difficult for people other than English speaking nations, because they do not know and cannot interpret a and b are abbreviation of apple and banana respectively.

(2) Use of symbolic expression

A. Wide use of symbolic expressions

Symbolic expression is used in almost all strands of mathematics in secondary level and above. In fact, it is widely used in all areas of algebra, elementary functions and analysis, analytic geometry, linear algebra, finite mathematics, probability and statistics and others. Speaking more correctly, without symbolic expressions it is impossible to develop quantitative reasoning in the areas. Symbolic expression is essential in sciences which use quantitative methods, and this can be seen typically in physical sciences and science of economics. This is also true for areas of technology.

B. Where symbolic expressions are effective

Symbolic expression is particularly effective in pattern-finding and generalization as well as problem solving in a broader sense, let alone study of functional relations (Bell 1995; DES & WO 1989; NCTM 1989, 2000; Royal Society & JMC Working Group 1995; Schoen 1988). Pattern-finding can be made by seeing that a situation is not unique but an element of a certain set. This assumes generation of variables, and to express with use of symbolic expressions is most appropriate for number patterns.

Example 1. Number pattern: Product of square plus one

For three integers 1,2 and 3, an equality

$$(1^2 + 1)(2^2 + 1) = 3^2 + 1 \qquad (1)$$

holds, as the left side is equal to $2 \times 5 = 10$ and it is equal to the right side. This equality seems to be a special and unique relation among three integers 1,2 and 3. However, we see another equality

$$(2^2 + 1)(3^2 + 1) = 7^2 + 1 \qquad (2)$$

holds. Thus we do not see a triple (1, 2, 3) as a special triple of three consecutive integers but we can see it as a triple of two consecutive integers and other integer. We can conjecture a pattern of the above form that for two consecutive integers a product of the sums of square of each integer and one is equal to sum of square of an integer and one. With use of symbolic expressions the conjectured equality is written concisely:

$$(n^2 + 1)((n + 1)^2 + 1) = A^2 + 1, \quad A \text{ is an integer.}$$

We find that the above equality holds when A is equal to $n(n + 1) + 1$.

Moreover, similar to the above equalities (1) and (2) we can find a pattern in

$$(1^2 + 1)(3^2 + 1) = 4^2 + 4,$$

$$(2^2 + 1)(4^2 + 1) = 9^2 + 4.$$

This leads to the following conjecture:

$$(n^2 + 1)((n + 2)^2 + 1) = P^2 + 4, \quad P \text{ is an integer.}$$

Generalization means "constructing variables" (Dörfler 1991). This presupposes existence of certain invariant. Generalization of propositions (Harel & Tall 1991; Lee 1996; Mason 1996) concerning numbers requires use of symbolic expression.

Example 2. Generalization of Example 1

From the equality $(n^2 + 1)((n + 1)^2 + 1) = A^2 + 1, \quad A= n(n + 1) + 1,$

in Example 1, we can make a generalization by regarding an operation + as variable and preserving forms of factors and result:

$$(n^2-1)((n + 1)^2-1) = B^2-1, B \text{ is an integer. } (B= n(n + 1)-1).$$

Moreover, we can go ahead by regarding 1 in factors and result as variable and preserving forms of factors and result:

$$(n^2 + k)((n + 1)^2 + k) = C^2 + k, k \text{ is any integer and } C \text{ is an integer.}$$

$$(C= n(n + 1)-k).$$

Further, when we consider n and $(n + m)$ in place of consecutive integers n and $(n + 1)$, we can find the generalized equalities hold:

$$(n^2 + 1)((n + m)^2 + 1) = Q^2 + m^2, \quad m \text{ is any integers and } Q \text{ is an integer.}$$

$$(Q=n(n + m) + 1).$$

$$(n^2 + k)((n + m)^2 + k) = R^2 + m^2k, k \text{ and } m \text{ are any integers and } R \text{ is an integer.}$$

$$(R = n(n + m) + k).$$

In problem solving, symbolic expression is used effectively not only for mathematically formulated problems but also ill-formulated ones. Typical in the former are applications of equation and inequality. For the latter, symbolic expression is used in problem formulation and its solution. Mathematical modeling is a typical example. Then specification of vital relationship among various quantitative relations in the situation is most important and symbolic expressions is suitable for the work.

Example 4. "Can Problem" (Kumagai 1999)

Among many kinds of cans, soft drink cans are focused on. They are assumed to be of cylinder shapes. Given below are data of their shapes, volume *V* (c.c.), radius of base circle *r* (cm), height *h* (cm) and rate *h/2r* :

Commodities	*V* (c.c.)	*r* (cm)	*h* (cm)	rate *h/2r*	(*s(x)*)
A	128	2.45	6.80	1.387	(1.011)
B	360	3.35	10.20	1.522	(1.019)
C	525	3.60	12.90	1.792	(1.036)
D	1242	5.15	14.90	1.447	(1.015)
E	2306	6.65	16.60	1.248	(1.005)
F	2528	6.80	17.40	1.279	(1.007)

If volume of a cylinder is constant, its surface area is minimal when $h = 2r$, or $h/2r = 1$, that is, when the height is equal to diameter of base circle. This is obtained with elementary calculus. A cylinder with minimal surface looks like a square when seen horizontally. As can be seen in the above table, shapes of soft drink cans differ largely from the optimistic one. Why does this happen? Or, why do soft drink companies disregard loss of surface materials and favor appearance and easy handling of cans? What follows is a simple solution of this question.

In a cylinder, whose radius of base circle is *r* and height is *h*, let *S* and *V* be its surface area and volume respectively.

$$S = 2\pi rh + 2\pi r^2, \quad V = \pi r^2 h.$$

Let *V* be constant value V_0. As said above, *S* is minimal when $h = 2r$. We express value of *r* and *S* when *S* is minimal as r_0 and S_{min} respectively.

$$V_0 = 2\pi r_0^{\ 3}, \quad S_{min} = 2\pi r_0 \times 2\, r_0 + 2\pi r_0^{\ 2} = 6\pi r_0^{\ 2}$$

Let *x* be *h/2r*. When *x* is equal to *1* , *S* is minimal and $S = S_{min}$.

Let $S(x)$ be surface area for *x*, and we consider $s(x) = S(x) / S_{min}$, which represents how far the surface area for *x* deviates from the minimal value. We will explore variation of $s(x) = S(x) / S_{min}$ as a function of *x* .

$$h = 2rx, \quad S(x) = 2\pi rh + 2\pi r^2 = 4\pi r^2 x + 2\pi r^2 = 2\pi r^2 (2x+1),$$

$$V_0 = \pi r^2 \times 2rx = 2\pi r^3 x = 2\pi r_0^{\ 3}, \quad r^3 x = r_0^{\ 3}, \quad r = r_0 x^{-1/3}$$

$$S(x) = 2\pi r_0^{\ 2} (2x+1)\, x^{-2/3}$$

$$s(x) = S(x) / S_{min} = (2\pi r_0^{\ 2} (2x+1)\, x^{-2/3}) / (6\pi r_0^{\ 2}) = 1/3 \times (2x+1)\, x^{-2/3}$$

$$= 1/3 \times ((2x+1)^3 / x^2)^{1/3} = 1/3 \times (8x+12+6/x+1/x^2)^{1/3}$$

Without knowing values of *s(x)* in interval $1.1 \leqq x \leqq 1.8$, however, we can neither know variation of *s(x)* in details nor reach to the solution of the question. Using spreadsheet (computer software) the following result is obtained:

x	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$s(x)$	1.001	1.004	1.007	1.012	1.017	1.023	1.030	1.036	1.043	1.050

Now it is clear that loss of surface material is unexpectedly small, e.g., for x less than 1.3 loss is below 1%. Values of *s(x)* corresponding to those of x in the table are given in the above table with parenthesizes. Therefore companies can disregard loss of surface materials and favor appearance and easy handling of cans.

We must not forget that only use of symbolic expressions cannot solve problems, but collaboration of computing tool such as calculator and computer is very helpful. However, as the above case, it is impossible for us to reach solution without use of symbolic expressions.

2 Teaching of Symbolic Expressions at Introductory Stage

In teaching of symbolic expressions in general, at introductory stage in particular, most crucial thing is not to produce illiteracy with respect to it as indispensable mathematical language. Since there is no doubt that symbolic expression is assumed to be newly introduced language, foreign language, the followings are required fundamental in teaching (Mochizuki & Yamada 1996):

- To let students have strong desire to communicate and to think with it.
- To make clear the purpose of communication and thinking.
- To emphasize what to communicate and think as well as rules of writing.

The proposal follows from the above perspective.

(1) Proposal for teaching at introductory stage

According to the principle that symbolic expression is foundation of mathematics for all but not for a few able, I would like to propose that teaching of it at introductory stage, in particular, is desirable to be organized so that language aspect and its use are combined systematically. The principle denotes that it is crucial to ensure solid foundation for all to be competent for using symbolic expressions as means of communication and thinking in mathematics. This may be interpreted immediately that students become competent to generate symbolic expressions and manipulate them. It is true only partially. I find an evidence that not a few students are reluctant to use symbolic expressions. Speaking in more details, even in a situation when use of symbolic expression is appropriate and students themselves are capable of using it, many of them are reluctant to use symbolic expressions and prefer to use numerical and/or geometric means instead of them.

The evidence comes from our experience of Japan-US Collaborative Research

(Nakahara-Ishida 1992), in which we surveyed an item "Pick a point P on the line segment AB and make two squares: One side of the one is AP and one side of the other is PB. Where should the point P be located to satisfy the condition that the sum of the areas of two squares is a minimum? Write a way of solution and the answer to the problem." I focused on result of Japanese subjects of 11th graders, who had learned algebra of quadratic equations and functions that is required for solving the item and were expected to be capable of using the necessary knowledge and skills. Eighty three percent of Japanese subjects answered correctly and the responses classified by the way they used are as follows:

Responses	Percent of responses (N=234)
• quadratic expressions (complete)	15 %
• quadratic expressions (incomplete)	13 %
• numerical tables (calculation of area)	26 %
• drawing figures	20 %
• explanation with words	15 %
• no answer	11 %

Only about one fourth, correctly 28%, of subjects used symbolic expressions including those who used incomplete ways. Except for those who gave no answer, the other 61% of subjects used numerical tables, drawing figures or explanation with words, which are only exemplary but not general.

Thus to be competent does not necessarily lead to the will to use, and becoming skillful at generating and manipulating symbolic expressions alone does not form a firm foundation of mathematics. Metaphorically, to become skillful is compared to training of grammar and spelling in writing. What to communicate and think and for what purpose are important as well.

Someone would say that the teaching advocated above is not different from what is implemented today. For instance, looking at Japanese Course of Study by Ministry of Education (MOE 1989, 1999), teaching content with respect to symbolic expressions in mathematics of the seventh grade, the first year of secondary school, contains generation of symbolic expressions, their computation (necessary for solving linear equation with one unknown), linear equation and its application, and direct and inverse proportion. This shows that students learn language aspect of symbolic expressions and its use at this grade and the teaching advocated seems to be present already. In actual classroom teaching, however, content are taught sequentially; to express first, to manipulate second and then to solve equations and word problems, and finally proportion. Neither systematic combination nor unification are made. In generation of expressions various kinds of quantities and their relations are taught. Student are involved in learning of these expressions and cannot afford to direct their thinking to what its use are. This reveals that appropriate consideration of contexts is absent. To express many kinds of quantities and their relations is considered to be aiming at preparation for solving word problems with use of equations and proportion in situations, because various kinds of quantities and their relations are contained in them. In computation of symbolic expressions, how to manipulate are taught and students engage diligently in practice to become skillful. For them to be proficient in the skills seems to be for the sake of itself or for good mark in mathematics class, though obviously moderate fluency of manipulation is necessary. After this long preparation, there appear solving linear equation as well as word problems with use of it and proportions. Can one imagine that all students persevere in long preparation of learning symbolic expressions? During preparation can they have any purpose of communicating and/or thinking? Can they have the will to communicate and think with symbolic expressions? Isn’t it true that what students learned and practiced are only rules and its application?

Thus crucial thing is teaching approach rather than scope of content in teaching. Desirable approach is the one which demolish the above mentioned sequential way and unify generation and manipulation of symbolic expressions - important language aspect of them - and their use systematically. Pattern-finding and generalization as well as problem solving are good examples of use of symbolic expressions (Bell 1995; DES & WO 1989; NCTM 1989, 2000; Royal Society & JMC Working Group 1995; Schoen 1988). Then suitable context setting must not be forgotten (Algebra Working Group to NCTM 1995). In addition, the followings are noted (Mochizuki & Yamada 1996):

- Let students appreciate symbolic expressions among various ways of representation, such as everyday language, numerical tables and graphs.
- Do not impose rules and restriction without reason and not force students to use specific expressions.

(2) Teaching of three processes in language aspect

Three processes, to express, to read, and to transform, form core part of teaching at introductory stage. It should be remarked firstly that letters in symbolic expressions at introductory stage are standing for numbers and numerical values. Secondly, from the viewpoint of form, symbolic expressions are classified into two, phrase type and sentence type. The former contains symbols expressing objects, signs of arithmetic operations and parenthesizes, and the latter is a form connecting two phrase type expressions with a sign expressing relation. Phrase type expresses quantity and sentence type expresses quantitative relation (Miwa 1996, 1998).

A. To express

Process to express, that is, to generate symbolic expressions is surely the start in teaching of symbolic expressions (Bell 1995; Royal Society & JMC Working Group 1995). Needless to say, everyday language structure as cultural background for students influences this process largely. Teacher must remember this importance always in their teaching.

In process to express in phrase type expressions, consideration from both side of signified quantities and signifying letters are needed. In the former, requisites are meaning of four arithmetic operations and various formulas among quantities as well as understanding of the order in integers and remainder classes by division. In the latter, following principles are implicit (Miwa 1996, 1998):

- Different quantities are expressed by using different letters.
- If a quantity is determined in terms of other quantities which were already expressed by letters and/or numerals, then the quantity is expressed in accordance with the way it is determined.

Here two problems are discussed. One is students' difficulties in generating symbolic expressions for quantities which they can calculate numerically. I guess considerable numbers of students have difficulty of this type. As they can calculate, they must know meaning of arithmetic operations and relations among quantities as well as formulas. They must be concentrating on calculation but unconscious of the process itself. Thus in order to overcome the difficulty, it is necessary to make conscious of the process in more details, that is, what quantities (and/or rates and so on) the numbers used in calculation stand for and how (by what kind of operations and in what order) they are calculated. Then, to make conscious of the process enables to reconsider the numbers like letters and to generate symbolic expressions. This attention to the process is related to what I discussed in the characteristics (d) of symbolic expression in 1 (1). For this object, to represent the process with everyday language and/or figures and diagrams is also effective.

The other problem is understanding the meaning of letters in expressions and their sequence in teaching. Usually, it is mentioned that meaning of letters includes definite but unknown numbers, generalized numbers and variables in functional relations (Küchemann 1981; Usiskin 1988). Therefore, teaching them systematically in an appropriate order is needed. If it fails, students must fall into serious confusion. For Japanese seventh graders, after teaching of linear equation teaching of direct proportion starts and expression $y = ax$ is presented. What is meant by letters x, y and a is explained to students but it may be uneasy task for them to discriminate letter x in equation and in proportion, let alone letter a.

I will add a brief explanation on process to express in sentence type expressions. Equation and equality are typical examples of sentence type expressions and made up by connecting two phrase type expressions with the equal sign"=". However, it is difficult for students to make up equation and equality generally as they are not mere connection of two phrase type expressions in many problems (Lockhead & Mestre 1988; MacGregor & Stacey 1993; Stacey & MacGregor to appear). Reversal error in Students-Professor Problem is an example. In making up an equation with one unknown, which is necessary in solving a word problem, through examining various quantities involved in a given problem and analyzing their relations, it is required to specify an equal quantity and to express the quantity in phrase type expressions with the use of letter x standing for an appropriate unknown quantity in two different ways (Bednarz & Dufour-Janvier 1994; MacGregor & Stacey 1996a ; Miwa 1996, 1998; Thompson & Thompson 1995).

B. To read

Process to read is primarily an inverse process of to express, that is, transition from symbolic expressions to signified quantities and their relations. It is to relate symbolic expressions to quantities and their relations and elucidate what symbolic expressions mean in the situation (MacGregor & Stacey 1994, 1996b). In reading, symbolic expressions are referred to everyday language, numerical values and figures according to the context of situations and user's intention (Miwa 1996, 1999). Yet we can read a symbolic expression in situations other than the original one. By this, what symbolic expressions mean can be enlarged into wider contexts. For instance, multiplication formula in algebra is read and interpreted as relation of areas or volumes geometrically.

Further, it is assumed that the process includes an arrival at deeper understanding and new vision out of which we can make a new discoveries and gain insight in the situation. In fact, generalization and specialization can be done by clear comprehension and reconsideration of what symbolic expressions mean (Miwa 1996, 1999). The following example is based on generalization idea:

Example 4 Discrimination of multiples of 17, 7, 13, 11, and 19

It is well-known that two digits number $10a+b$ is a multiple of nine if $a+b$ is a multiple of nine.

This is obvious seeing that

$$10a+b = (9+1)a + b = 9a + (a + b).$$

We consider 100 in place of 10 and some multiple of a prime number, e.g., 17, near 100, and preserve an idea of the discrimination of multiple of nine.

Let three and above digits number be $100a+b$, where a is a positive integer and b is a non-negative integer less than 100.

• Multiple of 17: As $102 = 17\times 6$ and $100a+b=102a+(-2a+b)$, $100a+b$ is a multiple of 17 if $(-2a+b)$ is a multiple of 17. For instance, 646 is a multiple of 17, because $a = 6$ and $-2a+b=-12+46=34=17\times 2$ in this case.

• Multiple of 7: As $98 = 7\times 14$ and $100a+b=98a+(2a+b)$, $100a+b$ is a multiple of 7 if $(2a+b)$ is a multiple of 7. For instance, 1001 is a multiple of 7, because $a = 10$ and $2a+b=20+1=21=7\times 3$ in this case.

• Multiple of 13, 11 and 19: As $104 = 13\times 8$, $99 = 11\times 9$ and $95= 19\times 5$, considering similarly we see that $100a+b$ is a multiple of 13 if $(-4a+b)$ is a multiple of 13, a multiple of 11 if $(a+b)$ is a multiple of 11, and a multiple of 19 if $(5a+b)$ is a multiple of 19, respectively.

This discrimination is useful when we know two digits multiples of these numbers.

C. To transform

Process to transform symbolic expressions is to change a given symbolic expression to a different form from the original one. For instance, in phrase type expressions, it is computation of expressions such as simplifying, developing and factoring (Bell 1995; Royal Society & JMC Working Group 1995).

To transform is done formally, that is, without considering the meaning of letters and the context of symbolic expressions. This is a remarkable strength of symbolic expressions. In compensation, however, transformation must be done under the transformation rules. The rules imply existence of invariant in transformation. In fact, for transformation of phrase type expressions the expressed quantities are invariant and numerical values of the expressions for any substitution are invariant if they are definite. For sentence type expressions quantitative relations are invariant (Miwa 1996, 1999).

As long as user obeys the transformation rules, transformation to any form is allowed. Therefore user cannot perform transformation effectively unless he is conscious of the goal. That is, transformation is guided by user's clearly intended goal. This is a partial but not minor reason why students often lose their ways in middle of using symbolic expressions and turn round and round uselessly (Miwa 1996, 1999).

In teaching of process to transform, transformation rules are desirable not to be given formally as rules to obey but be given as meaningful ways backed up with concrete things, such as numerical values, quantities and figures.

D. Relationship among three processes

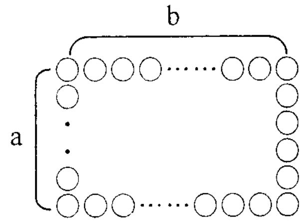
The roles of three processes in the use of symbolic expressions are sketched in Scheme of Use of Symbolic Expressions in 1 (1). Looking into details, however, three processes are considered not to be isolated but connected, and making good use of their connections is effective in teaching. What follows are a few examples (Arcavi 1994).

To read generated expressions and relate them to the original situation allows to check whether they express the quantities and their relations correctly or not. Thus to express and to read can be regarded as the processes of going back and forth. Substitution of numerical values in expressions is another example of check by reading. This is helpful for checking expressions generated from numerical tables (MacGregor & Stacey 1993).

Further, in selection of appropriate letters and expressions affording to represent structures fully in a situation, which is very important in process to express, we can examine whether the selection is good or not through reading the generated expressions. For instance, on a proposition "For three consecutive integers, square of the middle is one larger than the product of two other integers.", a student expressed $x^2 - yz=1$, expressing y, x and z for the three integers (Kieran 1992). To read this expression immediately makes clear that the expression is inappropriate because it does not express consecutiveness of integers which is an indispensable condition in the proposition.

In addition, to read often allows better selection of expressions. For instance, on a proposition "Sum of three consecutive integers is multiple of three." , a student expressed $n+(n+1)+(n+2)=3n+3$. We note that the right side is rewritten $3(n+1)$, and can read that the sum is equal to three times of the middle integer. When we express n for the middle integers, these three integers are expressed $n-1$, n and $n+1$, and their sum is equal to $(n-1) + n + (n+1) = 3n$. Then we can see symmetry in it and generalize the proposition easily by changing 1 to k, and by changing three consecutive integers to five and other odd consecutive integers (Miwa 1996, 1999).

Often different expressions of a quantity based on different ways of thinking appear in a situation and students question whether they are equal or not. Then transformation resolves the question. For instance, when marbles are arranged in a shape of rectangular sides whose length and width are a and b respectively, as illustrated in the figure in the right, the number of marbles on the sides in all is expressed in various ways. The followings are examples:



- $2(a+b)-4$, • $2a+2(b-2)$,
- $2(a-1)+2(b-1)$, • $ab-(a-2)(b-2)$.

By simple computation we can see easily they are all equal.

Whether transformation is performed correctly is checked by comparison of numerical values of transformed expressions with the original one by substitution. This is an example of check by reading (Miwa 1996, 1999). Further, to ascertain reasonableness of result by transformation is also done by reading, e.g., comparison of numerical values and drawing suitable figures.

I will add that reading makes it easy to determine the goal of transformation. For instance, when we wish to prove an equality (see Example 1 in 1(2))

$$(n^2 + 1)((n + 1)^2 + 1) = A^2 + 1, \quad A \text{ is an integer},$$

we see that by developing the left side,

$$n^2 (n + 1)^2 + n^2 + (n + 1)^2 + 1$$

must be equal to right side, $A^2 + 1$. Thus $n^2 (n + 1)^2 + n^2 + (n + 1)^2 = A^2$ is seen the goal of transformation.

(3) Relation to arithmetic teaching in elementary school

Obviously teaching of symbolic expressions is based on that of arithmetic in elementary school, as students' understanding of symbolic expression is based on that of arithmetic (Herscovics & Linchevski 1994; Kieran 1989, 1990; Lee & Wheeler 1989; Thompson & Thompson 1995). In fact, letters in symbolic expressions stand for numbers and numerical quantities, and operation signs and rules are same as those in arithmetic. However, there are differences between number expressions in arithmetic and symbolic expressions. For instance, in arithmetic concatenation of numerals represents commonly addition, such as $23 = 20 + 3$ and $2 \cdot 1/3 = 2 + 1/3$, while those in symbolic expressions represent multiplication, such as $2a = 2 \times a$, $ab = a \times b$. Moreover, students often recognize in arithmetic that left side of equal sign denotes process and right side product, such as $3 + 2 = 5$, but do not recognize that both sides are same or equivalent. It is not the case for symbolic expressions. Further, strategies in solving problems in arithmetic and those with use of equations - typical symbolic expressions - are very different (Kieran 1990, 1992). Thus many students' misconceptions and misunderstanding in symbolic expressions originate in their arithmetic learning. In teaching symbolic expressions, grouping what students leaned in arithmetic into two, those to preserve and strengthen and those to reconsider and revise, and taking both into consideration appropriately is essential.

What follows are those contained in the former and students are expected to master fully in arithmetic. They are very important in generation of symbolic expressions:

• <u>Number concept and its relation to arithmetic operations, particularly that of rational number and division</u> (OSGME 1987)

In arithmetic, what is needed is proficiency in four rules of numbers as well as grasp of closed property with respect to operation and making explicit expression of its result. Then division is crucial. In arithmetic, division of integers is done in three ways; to ask for quotient and remainder within integers, quotient as finite or infinite decimal fraction and quotient as fractional number. For instance, in division of 13 by 3, possible answers are 4 and remainder 1, 4.333••• and 4 • 1/3 (or 13/3). Some students are not confident that division of integers has a definite quotient. On the other hand, in symbolic expression $a \div 3$ is written in the fractional form $a/3$. Some students cannot recognize that $a/3$ stands for a definite number if a is not divisible by 3. Not only they may consider it something not understandable and cannot manipulate, but they may feel forced to write indefinite and mysterious thing due to convention in mathematics. This leads to distrustful feeling for symbolic expressions as typical tool of mathematics. In order to prevent such situation, teacher needs to help students recognize that division is always possible unless divisor is 0 and the result, quotient, is definite and expressed as fractional number. That is, students need to understand rational number concept and its expression a/b, where a and b are integers and $b \neq 0$.

In addition, to be understood is that for integers, multiplicative expression implies divisibility. For instance, that 15 is divisible by 3 is seen when 15 is expressed in a form 3×5, but it is unnecessary to do division 15 by 3 actually. This expression is used in multiple and divisor problems in symbolic expressions as well as arithmetic.

• Quantitative relations and their relationship to arithmetic operations

Geometric quantities, physical quantities and derived quantities from these as well as quantities in everyday life are taught in arithmetic. They are also treated in secondary mathematics. Relations of these quantities are also important. Quantitative relations are generally represented with four arithmetic operations and they are associated with meaning of these operations. Some of them are listed up in formulas with everyday language, e.g., speed is equal to length divided by lapse of time. Without understanding of quantitative relations represented with operations, generation of symbolic expression is almost impossible or very poor even if possible.

• Rate, ratio and proportionality (OSGME 1987)

It is acknowledged widely that many students have difficulties in generating symbolic expressions for fractional quantities, even if they can do for integral ones. It is also true for quantities and their relations in a problem which contains rate and ratio. Multiplication and division of numbers in general, those of fractional numbers in particular, and multiplicative relations between quantities are deeply rooted in an idea of proportionality. The idea obviously presupposes the notion of rate and ratio. These notions are supposed to be intrinsic but to make them aware is essential. Thus understanding of rate, ratio and proportionality is crucial in order to express quantities and their relations without restriction to integers.

(4) Most influential factors in teaching

Many factors influence mathematics teaching. Certainly most influential in teaching symbolic expressions is technology, let alone teacher (Fey 1989; House 1988; Kieran 1990; Wagner & Kieran 1989b). I would like to concentrate on technology and teacher here.

A. Technology

Generally speaking, technology stimulates and promotes students' various mathematical activities and helps them develop mathematical thinking. Calculators and computers, which are most familiar tools of technology in mathematics classrooms today, are very powerful. For instance, graphing calculators have ability of plotting graphs when expressions of functional relations are given and showing corresponding values in more details by zooming, and computers have ability of manipulation of symbolic expressions.

They are expected to be very helpful for teaching symbolic expressions (Kaput 1995; Royal Society & JMC Working Group 1995). However, I must point out that calculators and computers do not take the place of symbolic expressions, that is, when expressions are given the machines draw graphs and manipulates them, but they do not give the expressions themselves from problem situations. Moreover, it is obvious that they do not help students generate symbolic expressions directly. Though it is said that computer programming is of help for the generation of symbolic expressions (Sutherland 1989, 1991, 1993), it is another problem whether we can replace teaching of symbolic expressions with that of programming. I wonder if completely computer-dependent system of teaching of algebra, even if it exists, is effective for teaching of symbolic expressions. Another example of use of computer is drill and practice for generation and manipulation of symbolic expressions, and intelligent computer assisted instruction (or learning) is widely used now. This is considered effective and helpful to students. Obviously it is relevant to only limited part, drill and practice, of teaching of symbolic expressions but not to crucial aspects of it.

Thus today cooperation and concerted work of technology facility and teaching of symbolic expressions as well as thinking with use of them are most vital and desirable. I exemplified cooperation of computer facility and use of symbolic expressions in the "can problem" (Example 3 in 1 (2)), in which numerical investigation with spreadsheet and algebraic manipulation elucidated the reason why shapes of cans deviate from the optimal one.

Needless to say, as progress of information technology is very rapid and marvelous, what new mode of cooperation between information technology and teaching of symbolic expressions will be in the next century is surely one of the most crucial issues.

B. Teacher

Teacher is, as everyone knows, the most crucial factor in teaching. In teaching of symbolic expressions his/her role is particularly important (Thompson & Thompson 1995; Wagner & Kieran 1989b). One of the reasons is that symbolic expression is language. Teacher is acquainted with it but students are quite unfamiliar and they are strangers to it. Thus words used by students may have different meaning from teacher's words, and what teacher says may not be understood correctly by students and vice versa.

It is common that when students encounter symbolic expressions for the first time in mathematics class, they rely on their previous experiences in arithmetic learning. Students' previous experiences is surely helpful, but causes their misconceptions and misunderstanding actually, as I discussed in 2(3). As a proficient user of mathematical language, teacher is required to help students correct their misunderstanding, and to have sympathetic imagination for the students' misconceptions. Sometimes students' writings and actions may be beyond teacher's imagination. Teacher is expected to cope with these situations with perseverance. Examples of misconceptions and misunderstanding are found in mathematics education research studies in the world but it is clear that not all are found in them. In this respect, teacher as researcher is an expected role for him/her.

I will add another expectation for teacher regarding the role of example in teaching. In teaching of symbolic expressions, teacher often explains concepts and rules with use of examples. Important is concepts and rules in general and examples are used in order to facilitate understanding, and students are expected to see concepts and rules through the examples. This is also true in problem solving with use of symbolic expressions. Important thing is to know that use of symbolic expressions is very helpful for solving problems, including word problems. Frequently, however, in mathematics classes it is emphasized to understand given word problems which are worked out, while it is forgotten that these problems are examples.

Concluding Remarks

In the lecture, firstly I discussed characteristics of symbolic expression as mathematical language, and its wide use in mathematics and sciences and its effectiveness in pattern-finding and generalization as well as problem solving. Secondly I concentrated on teaching of symbolic expressions at introductory stage, presented proposal on the teaching and discussed teaching of language aspect, that is, three processes, to express, to read and to transform, and their connections. Then I considered its relation to arithmetic teaching in elementary school and finally addressed most influential factors, technology and teacher. These are relevant to school mathematics education but they are also helpful in lifelong education for adults and teacher education, I am sure.

Here I will mention a few issues on teaching of symbolic expressions at intermediate /advanced stage in school level, which I cannot address by the limitation of time:

・Understanding of meaning of letters extended beyond numerical quantities, such as vectors and matrices, whose algebraic structures are different from number field.
・Understanding of difference in meaning of letter as parameter and variable.
・Distinction of expression and its value, which leads to concept of identity.
・Becoming skillful in specifying vital relationship among quantitative relations in a problem.

Finally what I discussed today are certainly not new but well-known to many audience here, if not all. However, I will conclude that what is known is not always done well in classes and that there must exist further more for us to know in teaching and learning of mathematics in general, those of symbolic expressions in particular.

Note

1) This is a revised version with References of the text of a Regular Lecture in the 9th International Congress on Mathematical Education, held on 31 July - 6 August, 2000, at Makuhari, Chiba, Japan.

References

Algebra Working Group to the National Council of Teachers of Mathematics (NCTM) (1995). A Framework for Constructing a Vision of Algebra: a Discussion Document, Reston, VA, NCTM.

Arcavi, A. (1994). Symbol Sense: Informal Sense-making in Formal Mathematics, For the Learning of Mathematics, 14(3), 24-35.

Bednarz, N. et al. (Eds.). (1996). Approaches to Algebra, Kluwer, Dordrecht.

Bednarz, N. & B. Dufour-Janvier (1994). The Emergence and Development of Algebra in a Problem Solving Context: A Problem Analysis. In da Ponte, J. P. & J. F. Matos (Eds.), Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education (2), 64-71.

Bell, A. (1995). Purpose in School Algebra, Journal of Mathematical Behavior, 14, 41-73.

Coxford, A. F. (Eds.) (1988). The Ideas of Algebra, 1988 Year book, Reston, VA, NCTM.

Department of Education and Science (DES) and the Welsh Office (WO) (1989). Mathematics on the National Curriculum, London, Her Majesty ‘s Stationary Office.

Dörfler, W. (1991). Forms and Means of Generalization in Mathematics. In A. J. Bishop et al. (Eds.), Mathematical knowledge: Its Growth Through Teaching, Kluwer, Dordrecht, 63-85.

Dossey, J. (1998). Making Algebra Dynamic and Motivating: A National Challenge. In NCTM & MSEB (Eds.), The Nature and Role of Algebra in the K-12 Curriculum Proceedings of a National Symposium May 27 and 28, 1997, National Academy Press, 17-23.

Fey, J. T. (1989). School Algebra for the Year 2000. In S. Wagner & C. Kieran (Eds.), Research Issues in the Learning and Teaching of Algebra, NCTM, 199-213.

Harel, G. & D. Tall (1991). The General, the Abstract, and the Generic in Advanced Mathematics, For the Learning of Mathematics, 11(1), 38-42.

Herscovics, N. (1989). Cognitive Obstacles Encountered in the Learning of Algebra. In S. Wagner & C. Kieran (Eds.), Research Issues in the Learning and Teaching of Algebra, NCTM, 60-86.

Herscovics, N. & L. Linchevski (1994). A Cognitive Gap between Arithmetic and Algebra, Educational Studies in Mathematics, 27, 59-78.

House, P. A. (1988). Reshaping School Algebra: Why and How. In A. F. Coxford (Eds.), The Ideas of Algebra, 1988 Yearbook, NCTM, 1-7.

Kaput, J. (1995). Long-term Algebra Reform: Democratizing Access to Big Ideas. In C. B. Lacampagne et al. (Eds.), The Algebra Initiative Colloquium, U. S. Government Printing Office, 33-52.

Kaput, J. (1998). Transforming Algebra from an Engine of Inequality to an Engine of Mathematical Power by “algebrafying” the K-12 Curriculum. In NCTM & MSEB (Eds.), The Nature and Role of Algebra in the K-12 Curriculum Proceedings of a National Symposium May 27 and 28, 1997, National Academy Press, 25-26.

Kieran, C. (1989). The Early Learning of Algebra: a Structural Perspective. In S. Wagner & C. Kieran (Eds.), Research Issues in the Learning and Teaching of Algebra, NCTM, 33-56.

Kieran, C. (1990). Cognitive Processes Involved in Learning School Algebra. In P. Nesher & J. Kilpatrick (Eds.), Mathematics and Cognition, Cambridge University Press, 96 -112.

Kieran, C. (1992). The Learning and Teaching of School Algebra, In D. A. Grouws (Eds.), Handbook of Research on Mathematics Teaching and Learning, NCTM, 390-419.

Kieran, C. (1996). The Changing Face of School algebra. In Alisina, C. et al. (Eds.), 8th International Congress on Mathematical Education, Selected Lectures, S.E.E.M. ‘THALES’, 271-290.

Küchemann, D. (1981). Algebra. In K. M. Hart et al. (Eds.), Children’s Understanding of Mathematics:11-16, John Murray, 102-119.

Kumagai, K. (1999). Kyouzai Kaihatu no Siten : Sozai kara Kyouzai he - Kan Mondai no Kaihatu wo Tegakari nisite - (The Perspective for Development of Instructional Materials: From Raw Materials to Instructional Materials - with Development of “Can

Problem" as a Clue -. In Sugiyama, Y. (Eds.), Report of the Science Grant of Ministry of Education : Comprehensive Study on Curriculum Development Coping with Advanced Information Society, Tokyo Gakugei University, 25-37.

Lacampagne, C. B. et al. (Eds.). (1995). The Algebra Initiative Colloquium, Washington, D. C., U. S. Government Printing Office.

Lee, L. & D. Wheeler (1989). The Arithmetic Connection, Educational Studies in Mathematics, 20, 41-54.

Lee, L. (1996). An Initiation into Algebraic Culture Through Generalization Activities. In N. Bednarz et al. (Eds.), Approaches to Algebra, Kluwer, 87-106.

Lockhead, J., & J. P. Mestre. (1988). From Words to Algebra: Mending Misconceptions. In A. F. Coxford (Eds.), The Ideas of Algebra, 1988 Yearbook, NCTM, 127-135.

MacGregor, M. & K. Stacey (1993). Seeing a Pattern and Writing a Rule. In I. Hirabayashi et al. (Eds.), Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education, University of Tsukuba, (1), 181-188.

MacGregor, M. & K. Stacey (1994). Metalinguistic Awareness and Algebra Learning. In J. P. da Ponte & J. F. Matos (Eds.), Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education, University of Lisbon, (3), 200-207.

MacGregor, M. & K. Stacey (1996a). Learning to Formulate Equations for Problems. In Puig, L. & A. Gutierrez (Eds.), Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education, Universitat de Valencia, (3), 289-296.

MacGregor, M. & K. Stacey (1996b). Origins of Students' interpretations of Algebraic Notation. In Puig, L. & A. Gutierrez (Eds.), Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education, Universitat de Valencia (3), 297-304.

Mason, J. (1996). Expressing Generality and Roots of Algebra In N. Bednarz et al. (Eds.), Approaches to Algebra, Kluwer, 65-86.

Ministry of Education (MOE Monbusyo) (1989). Tyuugakkou Sidousyo Suugaku-hen (Guidebook for Lower Secondary School: Mathematics), Osaka, Osaka Syoseki,(in Japanese).

Ministry of Education (MOE Monbusyo) (1999). Tyuugakkou Gakusyuusidouyouryou Kaisetu Suugaku-hen (Explanation of the Course of Study for Lower Secondary School: Mathematics), Osaka, Osaka Syoseki. (in Japanese).

Miwa, T. (1996). Mojisikino Sidou: Josetu (Teaching of Symbolic Expressions : An Introduction) , Tsukuba Journal of Educational Study in Mathematics, 15, 1 - 14 (in Japanese).

Miwa, T. (1998). Teaching of Symbolic Expressions : An Introduction (1), Meiji Daigaku

Kyosyokukatei Nenpou (Yearbook of Teacher Education Program at Meiji University) 20, 47-56.

Miwa, T. (1999). Teaching of Symbolic Expressions : An Introduction (2), Meiji Daigaku Kyosyokukatei Nenpou (Yearbook of Teacher Education Program at Meiji University), 21, 15-25.

Mochizuki, A. & N. Yamada. (1996). Watasi no Eigo Jugyou (My English Lessons), Tokyo, Taisyuukan.(in Japanese)

Nakahara-Ishida, T. (1992). Nitibei Kyoutuu Tyousaniyoru Mondaikaiketu no Kenkyuu -“Seihoukeino Menseki “ nituite - (Study of “Area of Squares” in the Japan-US Common Survey on Mathematical Problem Solving). In T. Miwa (Eds.), Nihon to Amerika no Suugakuteki MondaiKaiketuno Sidou (Teaching of Mathematical Problem Solving in Japan and the U. S.). Toyokan, 119-134. (in Japanese).

Nakamura, K. (1962). Suugakusi (History of Mathematics), Osaka, Keirinkan. (in Japanese).

National Council of Teachers of Mathematics (NCTM) (1989). Curriculum and Evaluation Standards for School Mathematics, Reston, VA, NCTM.

National Council of Teachers of Mathematics (NCTM) (2000). Principles and Standards for School Mathematics, Reston, VA, NCTM.

National Council of Teachers of Mathematics (NCTM) & Mathematical Sciences Education Board (MSEB) (Eds.). (1998). The Nature and Role of Algebra in the K-12 Curriculum Proceedings of a National Symposium May 27 and 28, 1997, Washington, D. C. National Academy Press.

Osaka Study Group for Mathematics Education (OSGME Osaka Suugaku Kyouiku Kenkyuukai) (1987). Bunsuu・Mojisiki wo Osieru toiukoto (What It Means to Teach Symbolic Expressions and Fractions), Tokyo , Meiji Tosyo, (in Japanese).

Royal Society & JMC Working Group (1995). Teaching and Learning Algebra pre-19, London, Royal Society & JMC.

Schoen, H. (1988). Teaching Elementary Algebra with a Word Problem Focus. In A. F. Coxford (Eds.), The Ideas of Algebra, 1988 Year book, NCTM, 119-126.

Sfard, A. (1987). Two Conceptions of Mathematical Notions: Operational and Structural. In J. C. Bergeron et al (Eds.), Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education, Universite de Montreal (3), 162-169.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin, Educational Studies in Mathematics, 22, 1-36.

Stacey, K. & M. MacGregor : to appear. Curriculum Reform and Approach to Algebra. In R.

Sutherland (Eds.), Algebraic Processes and Structure, Kluwer.

Steen, L.A. (1995). Algebra for All: Dumbing Down or Summing Up? In C. B. Lacampagne et al. (Eds.), The Algebra Initiative Colloquium, U. S. Government Printing Office, 121-140.

Sutherland, R. (1989). Providing a Computer Based Framework for Algebraic Thinking, Educational Studies in Mathematics, 20, 317-344.

Sutherland, R. (1991). Some Unanswered Research Questions on the Teaching and Learning of Algebra, For the Learning of Mathematics, 11 (3), 40-46.

Sutherland, R. (1993). A Spreadsheet Approach to Solving Algebra Problems, Journal of Mathematical Behavior, 12, 353-383.

Thompson, A. G. & P. W. (1995). A Cognitive Perspective on the Mathematical Preparation of Teachers: The Case of Algebra In C. B. Lacampagne et al. (Eds.), The Algebra Initiative Colloquium, U. S. Government Printing Office, 95-116.

Usiskin, Z. (1988). Conceptions of School Algebra and Uses of Variables. In A. F. Coxford (Eds.), The Ideas of Algebra, 1988 Yearbook, NCTM, 8-19.

Wagner, S., & C. Kieran (Eds.). (1989a). Research Issues in the Learning and Teaching of Algebra, Reston, VA, NCTM.

Wagner, S., & C. Kieran. (1989b). An Agenda for Research on the Learning and Teaching of Algebra. In S. Wagner & C. Kieran (Eds.), Research Issues in the Learning and Teaching of Algebra, NCTM, 220-237.