Tsukuba Journal of Educational Study in Mathematics. Vol.20, 2001

ARTICLE

On dimensional analysis in development of multiplicative conception

Yutaka OHARA

Graduate School of Education, University of Tsukuba

This paper reports on an obstacle that students encounter in developing multiplicative conception. And the author tries here to make sure the historical process of it and propose two mental models by using ideas taken from dimensional analysis. Acting on these models, we analyze a process of solving two multiplicative problems posed to two secondary school students. Consequently, we recognize that it is helpful for us to take into account two points : i) the activation of dimensional analysis in developing students' multiplicative conception, and ii) the shift from unidimensional view to multidimensional view through verbalization (symbolization) of the new magnitude.

Background

It has been recognized that multiplication and division are important conceptions in current mathematics curricula. Multiplication and division problems that have dimensional complexity are more difficult than addition and subtraction problems that have only the unidimensional one. This epistemological complexity of the former problems has been revealed behind the mathematical simplicity. Over the past few decades, a number of studies have been conducted on this dimensional complexity of multiplication and division (e.g., Schwartz, 1988, Tompson, 1994). Although these studies have been reported that quantity with referents strongly constrain students' reasoning, little attention has been given to the competence to cope dimensional analysis. To investigate this problem, an analysis of student's activities is needed, as well as an analysis of historical processes. Such analysis may allow an explanation of some difficulties peculiar to multiplication and division.

This paper is intended as an investigation of two issues: 1) to point out controversial points in development of multiplicative conception, and 2) to exemplify a basic mental model for comprehending multiplication and division from the view of dimensional analysis. Section 2 describes an obstacle in comprehension of multiplication and division, as utilized the difference of multiplicative structures (Vergnaud, 1983). Besides it presents a Euclidean and Cartesians' view about multiplication and division in the history of mathematics and proposes two types of cognitive model as a framework to draw some aspects of shifting from scalar operation to functional operation. Finally, Section 3 reports a case study.

Dimensional issue in development of multiplicative conception

As many research studies point out, when students are requested to interpret multiplication and division, they emphasize not merely numerical aspects but also physical ones (e.g., Schwartz, 1988). Students usually deal with quantities that are not pure numbers but magnitudes of various kinds. Although a number is resulted from a process of abstraction through which it wrenches itself from any magnitude, it cannot exist without reference. Thus we cannot comprehend multiplication and division in view of the numerical framework alone. If we are to analyze any relationship in which students take into account, it requires dimensional framework that is derived from the product and quotient of homogeneous, and inhomogeneous magnitudes. In mathematics, dimensional analysis can be defined as dealing with the number of coordinates required to locate a point in space, and in this paper it's mean cooping the product or quotient of the fundamental magnitude raised to the appropriate power in a derived magnitude. In the matter of this framework, Vergnaud(1983) made several important statements. According to Vergnaud, we could identify three different multiplicative structures [1)]; (a) isomorphism of measure, (b) product of measures, and (c) multiple proportion. The isomorphism of measures is a structure that consists of simple direct proportion between two measure-spaces M1 and M2. The product of measure is a structure that consists of the Cartesian composition of two measure-spaces, M1 and M2, into a third, M3. What has to be noticed is that the product of measure is often less natural than isomorphism of measure[2)], because the way for analyzing and managing the latter structure seems to be by means of dimensional analysis. So it is difficult for students to master well this structure, and fail to understand multiplication and division when it is introduced as Cartesian product (Anghileri, 1989). Naturally, other researchers have mentioned the distinction between scalar and functional aspects (Freudenthal, 1978, Lamon, S. J., 1994, etc). It is easy for us to imagine the obstacle that students would meet in extending the meaning of multiplication from the "Isomorphism of measures" as a primitive conception to the "Product of measures".

The problem, which we have to consider next, is dimensional issue in history of mathematics. The above obstacle can be illuminated to consider historical parallel. Development of multiplicative reasoning would recapitulate in brief the whole history if mathematics could not be dissociated from its history. It is well known that Euclid had expressed the view in book V that the Greeks would only compare homogeneous magnitudes by forming their quotient (Heath, T., 1956). Euclidean Greek mathematics would not have been able to introduce a conceptual product PL, and conceptual quotient PL for two magnitudes P and L in general when P is one kind of magnitude (in one measure-space) and L is another kind of magnitude (in another measure-space). Essentially, it is due to the lack of conception of real numbers, but at the same time it is well known that Greek mathematics develop directly a mathematical theory of general physical quantities as in the fifth book of Euclid. So we would like to focus attention on metaphysical background of their reasoning. Greek mathematicians would envisage the proportion P1 : P2 = L1 : L2 if L1 and L2 are two values of the same magnitude such as lengths and P1 and P2 are two values of any other magnitude such as weights, they would not convert the proportion into an equality P1 : L1 = P2 : L2, or P1L1 = P2L2. While they had a prenotion of it surely, there was an obstacle in the metaphysical background of their reasoning which kept them from conceptualizing it in respect of dimensional analysis. They would not philosophize about the extension of the products and quotients to other magnitudes because they could not make sense of these extensions in their geometrical sense[3]). These issues made Greek mathematics unsuited to promote dimensional analysis in our sense. Given that we uphold above historical view, the mathematical conceptualization-cum-symbolization of physical dimension has been retarded by two thousand years.

Mathematicians divided magnitude by inhomogeneous one in the 17th century. It was expressly conceptualized during and after the Renaissance since Descartes in his geometry acts as if he spread the acute awareness of the role of symbolization by mathematically controlled reasoning and represents magnitude on the natural ordering of the one-dimensional linear continuum (Descartes, R., 1637/1973)[4]). He tried to emergent the general coordinates that represent physical magnitude in the natural linear ordering, and introduces this system into space that is Euclidean in current sense. When we venture to compare Greek mathematics with post Renaissance mathematics, we could give the following account. Greek mathematics intended to construct for a large class of magnitudes, and post Renaissance mathematics intended to construct the real number and envisages the isomorphism between the real number and the magnitudes of the large class. On the whole, Bochner(1966) has proposed plausible explanation as for the above. After pointing out that the Greek horror of multiplicative conception, he goes on to mention "Greek mathematics was both made and unmade by the efforts of the Greeks to conceptualize simple scalar magnitudes like length, area, and volume". As a matter of course, while we can take another historical view about Greek and Renaissance mathematics, I will take up above view as purpose of this paper is concerned.

On the basis of above view, the Cartesian product is historically difficult to be mastered. We should keep in mind that the mathematical conceptualization-cum-symbolization of a product like PL and a quotient P/L is one of merkmal for development of multiplicative conception. Student constructs notion that historically took centuries to evolve. Although we spend a lot of time coming to this phase, still we hope students are able to coordinate of distance, time, speed, acceleration etc, and to emergent new magnitude according to each situation.

An analysis of multiplicative structures provides for us only a framework for research, but they do not provide any description of students reasoning. It follows from what has been said thus far that it is better for us to introduce the frame of "mental model" as it is used in mental model theory (Johnson-Laird, P. N., 1983). As concerns mental model theory, it is possible for us to feature students reasoning. Mental models are the backbone of our reasoning process. They constrain activities in various ways. They could be a source of constraints on reasoning and problem solving process. In consequence, I venture to propose two types of mental model as follows (Ohara, 2000)

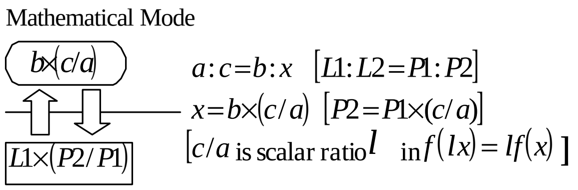


Figure.1. Mental model 1

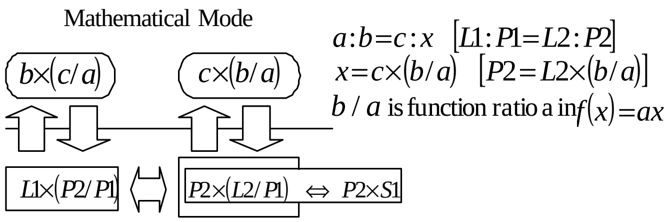


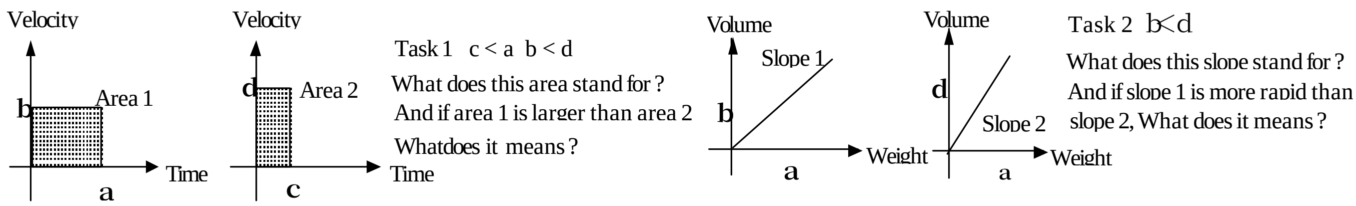
Figure.2. Mental model 2

Assuming that the students who have mental model 1 regard multiplication as a scalar operation modeling the increase of a quantity, and the students who have mental model 2 regard multiplication as a functional operation modeling the combination of two quantities. In figure 1 and 2, a,b,c,x are real numbers, and P1, P2 and L1, L2 and S3 belong to different measure spaces. In "mathematical mode", each student deals with a pure number that is unidimensional, and in the "physical mode", the student deals with a directed number that has dimensional complexity. Of course, each model is selected according to the student's aims, so it does not merely mean the extension of multiplication and division. The products PL, and the quotient P/L, are not only multiplies and divide the numerical values of P and L but it also forms a new magnitude out of the two magnitudes P and L. Thus, the shift between two mental models has profound implications for comprehending multiplicative conception that includes reasoning.

Case study

Method

Here, to actualize their mental models, and to illustrate how do students talk on the above issue, the problem solving activities given by Yumiko and Masahiro who are in secondary school students (14 year old) will be presented. 8 students' data were gathered in the early spring of 1999. They have already learned liner function, but this kind of tasks is nonroutine for them. They are asked to talk out loud while doing the following tasks. These tasks are modified from Schwarz's(1988) study. Although task 1 and 2 are actually constitutive ideas of the differential and integral calculus respectively, as for as the purpose of this paper is concerned, we would concentrate on students' dimensional analysis.



Results

Protocol of Masahiro. [Case 1.]

Session I (for task 1).

Masahiro wrote some numbers on the paper near the graph. Then he looked at the task 1and says: " no, wait....hm.....I have never seen it (time-velocity plane)." In session 1, there is no solution of continuity in time. (I: Interviewer, M: Masahiro)

I: Well, why are you in trouble ?

M: ...This... the left area is bigger than the right one, and.... a is bigger than b, and d is bigger than b...

I: Okay, How do you get that ?

M: Hm. wait..have a go to substitution, when a is 5,c is ...about 2, ...d is 5, ..(he wrote down 35,and 52)

I: Okay, well, what does this area stand for ?

M: Area, ah..um....here ? (laughing and pause)

Session 2 (for task 2).

Masahiro looked at the task 2, and began to argue that the slope 2 is steeper than slope 1.

M: Slope 2 is steeper than the slope 1, at a glance.

I: Hm...Okay, please explain it elaborately.

M: Well, ...I learned...this line (horizontal line) is x, this (vertical line) is y, so slope shows rate of changes!

I: two a are same in two graphs, so do you have something about b< d ?

M: Yeah...here...y is volume. ..so d is bigger than b.

I: What do these slopes 1 and 2 stand for ?

M: Rate of changes (in values of function)

Protocol of Yumiko. [Case 2.]

Session1 (for task 1).

Yumiko draws some numbers near a, b, c, d on the paper, and shows a bit of confusion and immediately change her approach. Then, she says "..um...so... so that... " (Y: Yumiko)

Y: So, if we multiply time by velocity, the area shows the distance. Since time and velocity are so related (co-variation), then we can multiply them.

I: What? What do you mean by "so related" ? please write it down on the paper if you need.

Y: (She wrote down the formula in words (velocity) = (distance) / (time)). So, concretely.... (she wrote down "3km/h4h =12km"), and I can reduce this fraction (she cut out "h").

I: Indeed, so, please tell me what ab < cd means ?

Y: ... well ... the right one might move more, the distance is longer than the left one, ... may be.

Session2 (for task 2).

Yumiko changes the sheet of paper and writes down some number near the graphs and says:

Y: This slope means.... That is.... rate of changes of function...

I: Okay, so what does this slope 1 and 2 stand for ?

Y: Slope...it is weight per volume, so...here (she wrote down the word expression (density) = (weight)/(volume)) , and...might be.. density... No !... (pause)...what do I call it ?

I: How did you get that ?

Y: Well..... dividing volume by weight to get one that.... look like density ...

Discussion

Session 1 and 2 were given to chart the presence of dimensional analysis in their reasoning. What two sessions make clear is that Yumiko and Masahiro had different meaning from an epistemological point of view. As a matter of fact, session 1 for them reveals the gap between their reasoning. In two sessions, Masahiro did not have a feeling of obviousness about the time-velocity plane, while he knew the coordination of time, distance, and velocity. It was hard for Masahiro to conceive area as distance. It is inferred from his trouble "what is area 1 and 2 in session1 that he avoid the epistemological obstacle (Sierpinska,1988, Mizoguchi,1992) caused by dimensional analysis. Not to mention, we have to discuss both epistemological and didactical nature that are two main causes for the beginning of obstacles. But they are too complicated to be examined in here. In session 2, while Masahiro was able to interpret task 2 and applied his reasoning to the two graphs, he would not mention the reciprocal magnitude for "density" that is coordinated from volume and weight when he compared the slope 1 with 2 . One interpretation of these things is that mental model 1 only allowed Masahiro to divide b by a depending on the isomorphic properties of the linear function so as to operate scalar ratio. His method of analysis would be constrained unidimensionally.

To the contrary, in session 1, Yumiko gave an account of area 1 and 2 writing the formula in words (velocity) = (distance) / (time), consequently "product of measure". Thus, she would regard multiplication as an operation to coordinate(generate) new magnitude in order to justify her conjecture, she analyzed dimension of "ab" and "cd" in session 1. What is more, in session 2 , she interpreted the "b/a" in physical mode, and tried to look for the new word that expresses new magnitude as a tool to compare two graphs in session 2. So it is reflected in her ability to coordinate another magnitude, and the process of "trying to name" is typically involved in mental model 2 (to output S1), and crucial in the process of dimensional analysis. Hence, Yumiko displayed an ability to deal with both dimensional and the numerical aspect during session 1 and 2. We can not disregard the fact that Yumiko do dimensional analysis in her reasoning that is required in mental model 2. Here the gap between Masahiro who have mental model 1 and Yumiko who have mental model 2 is quite evident. This raises the question of how to encourage their reasoning. To compare their reasoning brings us to the idea that it is hopeless to try to propose better conditions for students to make sure of their mental model if we do not make the effort to get them to interpret graphical representations of relationships between the three magnitudes.

Concluding remarks

I would state that it is essential for us to take into account the dimensional analysis as well as the analysis of number systems when we have students comprehend multiplication and division.

From what has been discussed above, we could conclude as follows : firstly, it is helpful for us to dare to activate dimensional analysis in students' multiplicative reasoning to shift from unidimensional view to multidimensional one. A way to actualize it and get the student to shift between mental models might be to use graphical representations for which need dimensional analysis. Secondly, it is adequate to encourage student's multiplicative reasoning by verbalizing (symbolizing) that generates new magnitude. Students' reasoning advance only if they encounter situations that they fail to assimilate, and then teachers are able to help students to accommodate their views. While it is right in a way to deal only with numerical aspects eventually in mathematics, it is better for us to consider the dimensional analysis as a key to develop students reasoning. To explore how to encourage them practically remains as a matter to be discussed further.

Notes

1) The third structure "multiple proportion ", that is not canonical way of choosing unit as f(1.1)=1 has similar relationship to "Product of measure"(Vergnaud,1983), so we do not deal with this third structure in this paper.
2) We are able to see"product of measures" as a double isomorphism or double, and reciprocally "isomorphism of measures" can be explained as a product such as volumedensity = mass, in short it depends on our interpretation.
3) There were two exceptional cases for P and L. First, P and L are both the lengths, and their product is an area. Second, P is the length and L is the area, and their product is a volume.
4) In strictly, this method was traditional one since Merton school, and Fourier,J.(1822) had been given systematic method to coordinate physical dimension.

References

Anghileri, J. (1989). An investigation of young children's understanding of multiplication, Educational Studies in mathematics **20**(**4**), 367-385.

Bochner, S. (1966). The role of mathematics in the rise of science, Princeton University Press.

Boulet, G. (1998). On the essence of Multiplication. For the Learning of mathematics **18**(**3**), 12-19.

Boyer, C. B. (1956). History of Analytic Geometry, New York : Scripta Mathematica.

Clagett, M. (1966). The science of mechanics in the middle ages, Madison.

Freudenthal, H. (1978). Weeding and sowing: Preface to a science of mathematical education, Dordrecht Boston : Reidel.

Descartes, D. (1637/1973). Discours de la methode, Osnabrueck : Editio Simile.

Johnson-Laird, P. N. (1983). Mental models: Towards a cognitive science of language, inference, and consciousness, Cambridge University Press.

Fauvel, J. (1988). Cartesian and Euclidean Rhetoric, For the Learning of Mathematics **8**(**1**), 25-29.

Kaput, J. (1985). Multiplicative word problems and intensive quantities, Harvard University, Educational Technology Center.

Lamon, S. J. (1994). Ration and proportion: Cognitive foundations in unitizing and norming. In

G. Harel & J. Confrey (Eds.), The Development of Multiplicative Reasoning in the Learning of Mathematics (pp. 89-120), State University of New York Press.

Heath, T. L. (1956). The thirteen books of euclid's elements vol.1, Cambridge: Cambridge university press.

Mizoguchi, T. (1992). A hypothetical model for overcoming epistemological obstacle: focus on the notion of limit, Tsukuba Journal of Educational Study in Mathematics **11**(**B**), 37-48.

Ohara, Y. (2000). Epistemological complexity of multiplication and division from the view of dimensional analysis, Proceeding of History and Pedagogy in Mathemaitics, Taipei, 189-195.

Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & Behr (Eds.), Number concepts and operations in the middle grades (pp. 41-52), NCTM.

Sierpinska, A. (1988). Epistemological remarks on functions, Proceedings of the Twelfth International Conference For the Psychology of Mathematics Education, Vesprem, Hungary, 568-575.

Thompon, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), The Development of Multiplicative Reasoning in the Learning of Mathematics (pp. 181-234), State University of New York Press.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau. (Eds.), Acquisition of Mathematics Concepts and processes (pp. 127-174), New York; Academic Press.

* This paper is a renewal version the paper presented on the International Conference of History and Pedagogy of Mathematics which was held on August 14, 2000. Taipei.