Tsukuba Journal of Educational Study in Mathematics. Vol.20, 2001

REPORT

LEGO Project[1)]
Mediational means for Mathematics by Mechanics

Masami ISODA[2)],
Akira SUZUKI, Yutaka OHNEDA, Masahiko SAKAMOTO, Naohito MIZUTANI[3)],
Nobuaki KAWASAKI[4)], Tatsuo MOROZUMI[5)],
Shigeki KITAJIMA, Norifumi HIROI[6)], Kazuhiro AOYAMA, Akio MATSUZAKI[7)]

LEGO Project[8)], which begins in 1996, is a part of the research project 'Algebra, Geometry and Calculus for All' since 1993. Algebra, Geometry and Calculus for All project has developed the secondary level curriculum material to support students getting higher mathematical concepts through inquiry with the innovative use of technology. This paper describes the significance of the LEGO Project in which mechanical structures made with LEGO will represent madiational means of higher order mathematical concepts. We will also discuss the strong effects of using LEGO mechanism such as object, subject and method for mathematical inquiry in the case of transformation at secondary level.

Introduction

In this research project, LEGO mechanics are used as the mediational means of higher order mathematical concept. The reasons why we use LEGO mechanics at the age of innovative technology are as follows; First, higher order mathematical concepts could be mediated by mechanics. Second, innovative tools such as computer are not enough and integrated use of traditional and innovative tools is needed. Third, integrated use of tools innovate mathematics education from the view point of representations. As the introduction, we will discuss these three points for the rationale of the LEGO project. Then, we will discuss some specific features of project topics; LEGO mechanics give students the subject, method and object of inquiry which interest students; LEGO mechanics offer analysis before proof; Through the process of making mechanics, LEGO mechanics are internalized; and, LEGO mechanics are seen as representation tools for mathematics.

Rationale

Higher Order Mathematical Concept Mediated by Mechanics

Figure 1 is from Descartes (1637) and Figure 2 is from a Japanese textbook (1943) for the secondary level mathematics. Until the age of modernization (New Math), every teacher used to use such kind of mechanics (mechanisms or tools) but in these twenty years, the innovative using of new technology such as computer has become the center of attention and mechanical tools were scrapped at spring cleanings. Today, many of new-age teachers do not have any experience using those mathematical tools. But we should know that we could not easily replace traditional tools with new technology such as computer.

For the discussion of this point, the Vygotskian perspective (Wertsch 1991) or socio-cultural perspective (Otte, 1991) about meditational means which include all of tools such as physical tools like mechanics and psychological tools like mathematical representation are useful. The socio-historical-cultural perspective describes that *each mediational means has embedded own historical-cultural functions and restrictions* (Wertsch 1991).

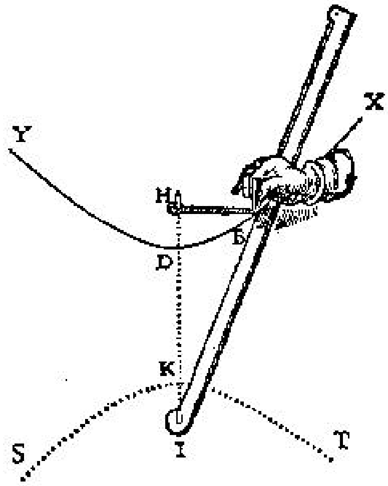

Figure.1 Descartes 1637

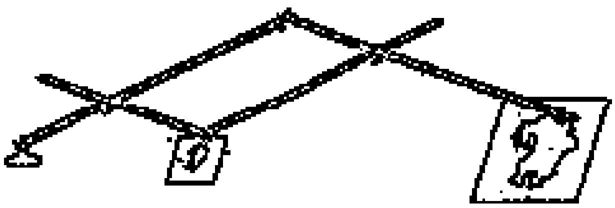
Figure.2 Japanese Textbook 1943

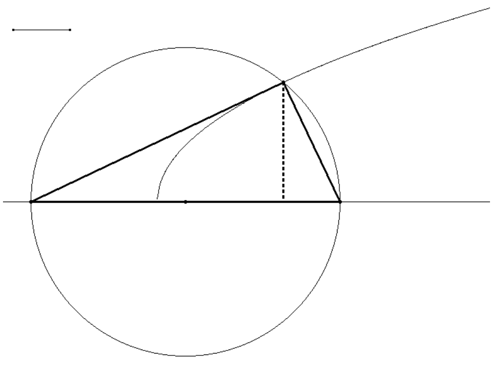
Figure. 3 via Geometric Mean

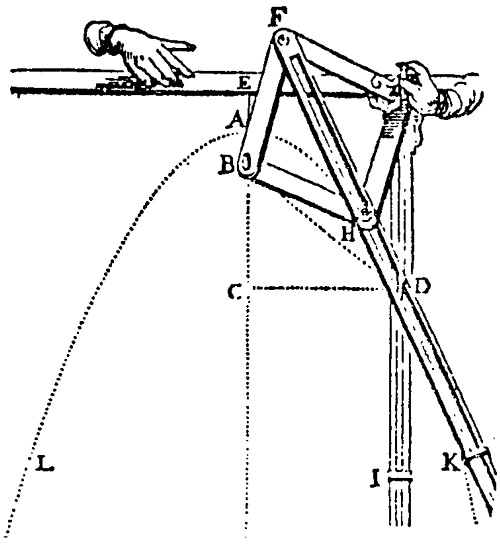

Figure. 4 Schooten 1646

For example, figure 3 is the locus given by the geometric mean, the dotted line in figure 3, used in the well known right triangle about the geometric mean at the junior secondary level in Japan. Two sides meeting at the right angle represent the tangent line and the normal line of parabola. Figure 4 (Schooten 1646, from Maanen 1991, Denis 1995) is a mechanical device for drawing parabola. Before drawing a parabola, tangent of the parabola was already given as the rod of mechanism. Because the ancients obtained loci using tangents and normals, the makers of calculus tried to find a ways to obtain curves from given tangents and normals (i.e. integration), and the new way of getting tangent and normal against any given curves (i.e. differentiation). Today, we draw curves as the graphs of equations. By representing curves with equations, we lost the object of questions such as the relation between curves and tangents.

This historical truth supports not only the importance of referring to the embedded functions and restriction (limitations) of mediational means but also the merit of using tools; tools give us the *subject* and *method* of inquiry. Indeed, because Descartes had a metaphor of mechanics he could inquiry this relation and because the ancients used to inquire geometry with constructions on sandboxes they inquired about curves with tangents and normals. Tools also give us the *object* or *context* of inquiry. For example, historically, to *design* or *make* a mechanical device which changes circular motion to linear motion was a good problem of mechanics and it could be proved by that of well known triangle in figure 3 (see Kempe, 1877, Smith 1959).

From these historical points of view, we could argue not to replace mechanics with new technology because if we do, we might lose these points, easily.

Educational Tools or Representation Tools

In didactics, there were traditional discussions about the function and importance of educational tools or aids as specialized tools in school education. But if we discuss tools from a historical perspective, those traditional discussions in didactics which focused, focused on the activity in the classroom look strange because some of the tools were invented based on historical inquiry in academic research context. These tools are embedded in mathematics itself, and the reason why we prefer to use any tool in classroom should be that it is necessary for inquiry of mathematics and by using the tool, we can enhance students' *historical-cultural appreciation* as well as *appreciation of mathematical ideas and representations*.

Today, in the age of technological innovation, there are two strong reasons why we could not easily use traditional word of 'educational' tools. First, the technological innovation focused on the importance of the diversity of representation and did not focus on the specific representation. On the other hand, the word 'educational tool' is focused a specific function of tools to teach something. In the teaching context, a tool could be seen as a model of mathematical concept in classroom. In this context, if we could get an easier tool to teach it, we throw it a way. We already experienced it at the age of new math. At the new math, abstract concept is important and some focused tools were only used to transform but other tools were thrown a way. Because in that age, focused tools were educational but defocused tools were not educational. Today, abstract math is operated with computer as well as numerical calculation. We should prefer the appropriate representation for problematic situation and we could not fix some specific relation such as ordered relation between abstract math and a model. Second, in the case of computer, we use the same tool for the educational aim and academic research. We can access any information which could not be restricted to educational aim. In this border less environment, we can not distinguish educational tools from the other tools any more.

From the view point of maditational means in mathematics, each tools or representation has restriction (merits and demerits). *In the age of technological innovation, most desirable ability is to detach limited representation, select a better representation and in some case, make or design an appropriate representation*. In this context, better way to use tools is to integrate many tools such as physical tools, computer tools and mathematical representations. Moreover, variable features of tools such as *designability* and *decomposability* are important. Indeed, today's computer software has the macro function which could be seen as a decomposable feature.

Background Issues and Beyond limitations

During the age of modernization (new math), mathematics education issues for innovation focused on teaching more abstract or higher mathematics because abstract mathematics is absolute and applicable. Today, most of mathematicians use a computer algebra system (CAS) for their research and many of them call their computer room as 'Lab'. Most users of mathematics usually use such tools without knowing mathematical theory behind them. Even if leaner do not have appropriate understanding of higher mathematical concept, he/she can use and explore mathematics with computer in some extent. These today's revolutionary issues enhance the relativism of mathematical inquiry. For example, Japanese new junior secondary mathematics curriculum took in the "observing, operating, and experiment" for getting mathematical knowledge as well as mathematical reasoning and proof. This new curriculum enhanced that mathematical proof is not only the way to accept the mathematical fact.

In this context, the validity of mathematical knowledge is strongly restricted by meditational means such as tools and representation. On this issue, important point is not only absolute authority but also our assuredness and knowing the limitation of it. For example, when we get the same result with accustomed tools or ones representations, with several diversities, we feel deeper appreciation. This appreciation is a better emergence of restriction.

On the other hand, there are negative restrictions, limitations, in each tool. Thus, we have to overcome such restrictions through changing tools or representation depending on what we need. Following example (Masami Isoda, Akio Matuzaki, 1999) demonstrate how changing tools and representations help students to overcome the restriction of each tool and representation.

Students were asked to make the mechanism of wooden-horse on a merry-go-round. They made a horse itself because their reasoning with visual images was based on crank mechanism. In this situation LEGO was mediational means to represent their experience. At the next stage, students were given the crank mechanism made with LEGO. They observed motion of the mechanism but they explained that the motions of horse were similar circles at the beginning and they needed some discussion before they understood the motions were 'egg curves' which had the same amplitude. Even after they operated mechanics, they still had some restrictions until they could reason with the real structure. At the third stage, students asked to represent the piston motion of crank by the equation of function. Students drew the graph of the function with Graphic Calculator and recognized that the equation of the function represented the piston motion well. But when students got the non-continuos graphs in the special cases of the function, they thought their equations were not appropriate for the mathematical representations of mechanical structure because they were reasoning with the mathematical representation without non-mathematical structure. Students have some cognitive limitation for reasoning with algebraic representation. At the forth stage, students were asked to make the crank by LEGO and they could interpret the meaning of non-continuos graphs. They could recognize the equation of function represented the real structure well and we could say that they could reason with mathematical structure corresponding to the mechanical structure.

In this example, we could observe several cognitive obstacles coming from the restriction of mediated means and students overcame these obstacles by using and changing tools and representations. Inquiry based on tools and representations enhances the relative future of knowledge construction, and it calls for the integration of using tools and representations for getting valid knowledge of mathematics.

LEGO Is Better Hands-on Goods

For representing mechanisms of mechanical tools, this research project preferred LEGO because students can make mechanism easily and they have already a lot of experience using LEGO. There are three roots for the LEGO project; First, there are many studies related to the use of mechanical tools. In Japan, there were good textbooks including mechanics at secondary level mathematics in 1943. Second, there are remarkable researches which uses mechanics and new technology with historical perspective in US by Jere Confrey and David Dennis and in Italy by Maria B. Bussi. In particular, Jere Confrey already used LEGO mechanics. The major difference between the LEGO project and Maria's research using mathematics machine is that the LEGO project sees mechanics as decomposable and includes constructing activity in mathematical activity. Third, in educational technology area, there are some researches about LEGO mechanics and LOGO programming. Maker or Dealer could support the project.

Subject, Object and Method of Inquiry

LEGO mechanics gave students subject, method and object of inquiry depending on their interest. Pantograph is well known mechanics for teaching similar figure and it exemplify these essential features of LEGO as mediated means for mathematics. Following lessons were done by Akira Suzuki at the Junior High School attached to the University of Tsukuba.

The Lesson of Similarity with the Pantograph of LEGO

Similarity of shapes is taught in the 8th graders, 2nd grade of junior high school in Japan. The following lesson was for teaching of similar figure using pantograph by LEGO (figure 5).

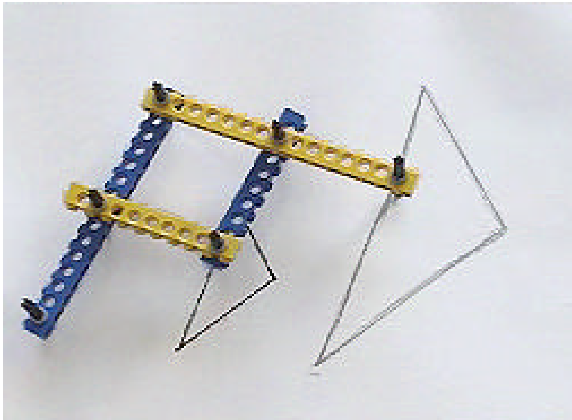

Figure 5. Pantograph by LEGO

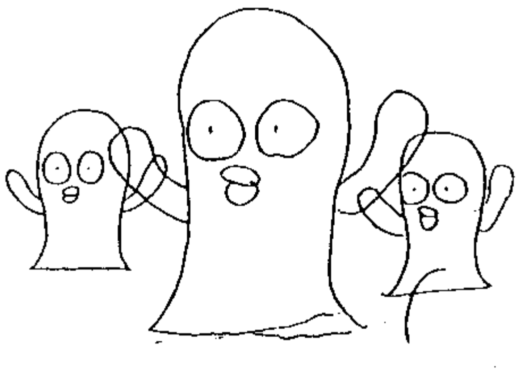

Figure 6. The example by a student

Lesson by Akira Suzuki

Modes of Teaching	Teaching Activities and expected students' reactions	Remarks
Explanation	Explain the function of pantograph. Ask for any question and motivate their interest. Explain that it is a tool to draw similar figures and how to use it.	Demonstrate and convince how to use pantograph made of LEGO.
Give students First Task Students' Operation Activity	Task 1. "Draw any figure on a paper and draw a similar figure using pantograph." Give all of the students a pantograph made with LEGO and a paper, and let them draw any figure. As students draw, they should become interested in why the pantograph draws a similar figure.	Students' pantographs are also made of LEGO because it also have to be decomposable. 'Decomposable' is important to change parts and parameter.
Illustrate Second Task Draw a pantograph on the board and inquire the reason Explanation	Task 2. "Compare figures that you have drawn originally and by using pantograph. Through finding the principle of similarity, let's consider the reason why pantograph could draw similar figure." Question: What properties are there? Students: The square in there is always a parallelogram. Question: How are the positions of the original figure and the one drawn with pantograph related? Student's answer: Similar. Students explain the reason.	In order to search the reason, illustrate pantograph on the board. To discover hidden property, illustrate the center of similar figure and the position of similarity. Explanation was done by students' reasoning

Merits of Pantograph

There are at least two preconceptions for the definition of similarity when students learn similarity;

1. Enlargement or reduction of a figure is similar to its original figure.
2. Two polygons with proportional corresponding sides and equal angles are similar.

Students learn the first point in elementary school, enlargement or reduction. We should also confirm students second point for the definition of the position of similarity as well as the idea of first point. On the other hands, students may get annoyed if they have to draw similar polygons using ruler and compasses, and it takes large amount of time. Thus, at the introduction of similarity, teachers usually gave students similar figures (polygons) that are already drawn on a sheet of paper or the textbook and tried to induce the second point for defining the position of similarity. Even after this expanding definition, students do not have a chance to draw figure containing curved lines.

Thus, it is anticipated that if we gave students an easier tool for the enlargement and reduction, the more students would experience enjoyable, familiar and interesting activity in similarity. This will, in turn, promote better understanding. Pantograph is not only tools for similarity. Computer and Xerox machine can easily make similar figures, though, they are not so challenging because students already experienced it at elementary school. On the other hand, the classical enlargement, reducing tool, pantograph, has miracle feature which enables us to draw similar figure easily, even a figure containing curved lines. In addition, pantograph can provide the ground for explaining the reasoning why the drawn figures are similar.

A pantograph is a tool that student can use while grappling with problem with interest and independently through activities. Recently, diversity through the use of technology in education has been spread; however, a pantograph is a real tool, not virtual one like computers. Also a pantograph itself possesses mathematical properties. Students would rise their interest just drawing a similar figure with pantograph.

The use of pantograph enables us to enlarge and reduce a figure with curved lines, which was not possible before, and it also mediates the first point abouve. Pantograph also realizes virtual machine of the transformation of similarity.

At The task 1, pantograph was a way of drawing similar figure at first but at the same time, letting students draw a similar figure gave students the subject of inquiry for investigating the principle of similarity and structure of the pantograph which enable students to draw similar figure. Because we used LEGO for pantograph, students could easily change the structure of pantograph. Thus, students could know the importance of the structure, which enable them to draw similar figure. At the task 2, students could realize the necessity of proof because there is not any promise that their drawn figures are similar. Inquiring and explaining the structure of pantograph are now the object of study. Through the explanation of reason, the center of similarity and its position could emerge as the subject to be taught.

Students Response

The Junior High School attached to the University of Tsukuba owns pantographs for a couple of decades. There is no difference with other teaching methods if the pantograph is just used to demonstrate and lead to the definition or theorem. In the class, students were concerned or felt interested via experiencing the drawing with LEGO pantograph. Each student should have a chance to operate his/her own pantograph.

In the class, students were paired, and they drew whatever figure they want, such as cartoons (figure 6) and letters (characters). Students wondered why it draws a similar figure, and started inquiring. Based on self-inquiry of students, the class smoothly went with what teacher planned. In the class, teacher let students demonstrate the pantograph, list properties, and consider the reason. In this case, before this lesson, students already learned first point, second point and definition of the position of similarity in one lesson and experienced to draw similar figures with ruler and compasses, they easily find the position of similarity and explain the reason.

According to students' impression, pantographs are accepted very well among most of them; some of their impressions are "pantograph is fun, we learned through playing", "ancients who invested pantograph is wonderful", and "the class was fun, too".

The pantograph made students interested in similarity. Another merit of it was its composition of LEGO. Students deconstructed and reconstructed it and drew various figures including non-similar figures. Making a pantographs with adequate materials such as LEGO in class would bring students more interests.

Analysis by Mechanics

Sylbester's Pantograph

Sylbester's pantograph (figure 7) which is made from parallelogram OABC and two similar triangles, APB and CP'B, is the mechanism which could draw rotated similar figure. Maria (1993, 1996) researched the effect of it for shifting the levels of proof. The following example also exemplifies the effect of it for proof in Japanese Case from the view point of the inquiry of mechanism. In Japanese junior secondary curriculum, geometric proof is taught from grade 8 but this lesson was done in grade 7, 1st grade of Japanese junior high school. Students do not know formally the conditions of congruent triangles but learned the conditions of determining a triangle at elementary school. To reduce difficulty, the teacher, Masahiko Sakamoto, planed to use LEGO pantographs in the case of congruent equilateral triangles for similar triangles (figure 8). Students already know various angle relationships in the case of triangles and parallel lines but their knowledge about symbolic representations of figures is still weak.2

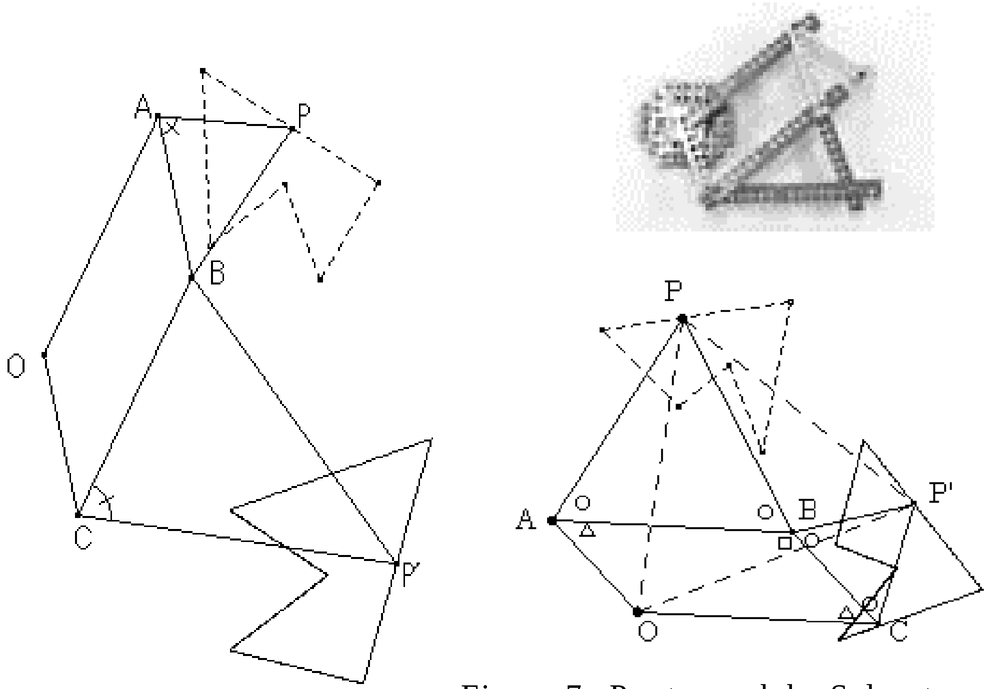

Figure 7. Pantograph by Sylvester

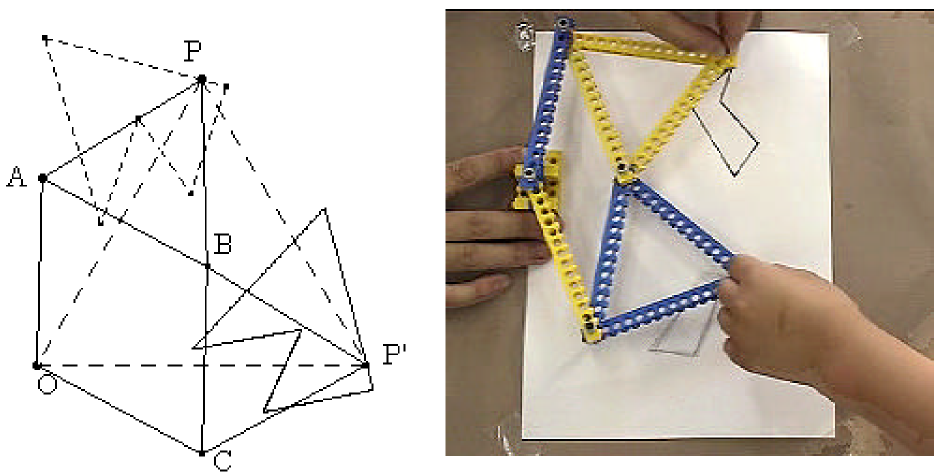

Figure 8. Demonstration

By using LEGO pantographs, students discussed the way of constructing parallel translation, line symmetry and symmetry with respect to a point by ruler and compass for about one hour, and the teacher confirmed the symmetry with respect to point is rotation of 180 degrees. In the second lesson, teacher asked students how to construct the rotated figure other than 180 degrees. The teacher showed a LEGO pantograph and told students that it could draw the figure rotate 60 degrees. Teacher did not demonstrate how to use it and gave each group time to explore. Students explored for 20 minutes. Some groups found the way to rotate

60 degrees but many groups could not find the point of fulcrum to rotate and could not determine the way of a 60 degrees rotation. Students explained their ways. After they knew how to draw, they continued to explore and inquire the reason and found invariant properties which kept the pantograph as follows (figure 8).

○＝60°,
△＋□＝180°,
△POP' must be equilateral
Angle POP' must be 60? Angle PBP' must be ○＋△.

At the end of the second lesson, students knew that they had to explain angle POP' is 60°, or triangle POP' is equilateral. To construct the explanation was their homework. During the third lesson, students explained their reason .

Explanation by some students

'If 360°－(2×○+□)=△＋□, then angle PBP' equals angle PAO and angle P'CO. Thus, if we rotate △PBP' 60° at point P, then △PBP' comes to △PAO and if we rotate △PBP' 60° at point P', then △PBP' comes to △OCP'. Then △POP' is equilateral. Thus, we should explain 360° (2×○+□)＝△＋□.'

360°－(2×○＋□)＝360－2×60°－□=240°－□
△＋○=△＋60°=(180°－□)+60°=240°－□, Thus 360°－(2×○+□)＝△＋□.

One of students who came up with explanation 1 described his reasoning as follows;

'Teacher told us "there are angles which keep same angle if you manipulate LEGO mechanics." Thus, I manipulated mechanics to find angles which kept 60 degrees or same degree.'

His way of findings is just 'analysis by mechanics' because he found his explanation beginning from the assumption of the result of 60 degrees. Ancients used the ward of analysis to find the way of proof by beginning from the result. Archimedes discussed his way of analysis as the methods of heuristics by mechanics because his methods of getting area began by supposing the balance between figure and segment which represent the area of figure. Students' reasoning could also be called 'analysis by mechanics' because LEGO mechanics enabled him to assume the result.

After Explanation

Throughout the class, students responded very well. After the lesson, students wrote their impression as follows;

'I wonder that LEGO could be a drawing tool', 'First question, how could we use this tool was unexpected and we could feel interest from outside of mathematics but we knew the question was also mathematical.', 'Mathematics is theory for everything', 'For getting expressions, we used the result but, after that, we calculated the expression to derive the result.' 'We learned a new perspective for problem posing and solving'

Each student's impressions exemplifies how the pantograph could function as the mediational means for mathematics. The pantograph has meanings more than transformation machine; One of its other meanings is the means for analysis.

Internalization of Mechanism with the Context of Making

The lesson after the Sylvester's pantograph is to design the mechanism with LEGO which transform the figure by line symmetry such as shown in figure 9. At the beginning of the lesson, teacher demonstrated the way to use Sylvester's pantograph to rotate the figure 60 degrees for confirming the existence of mechanics to draw the transformed figure and asked students to design the mechanism to transform figure by line symmetry.

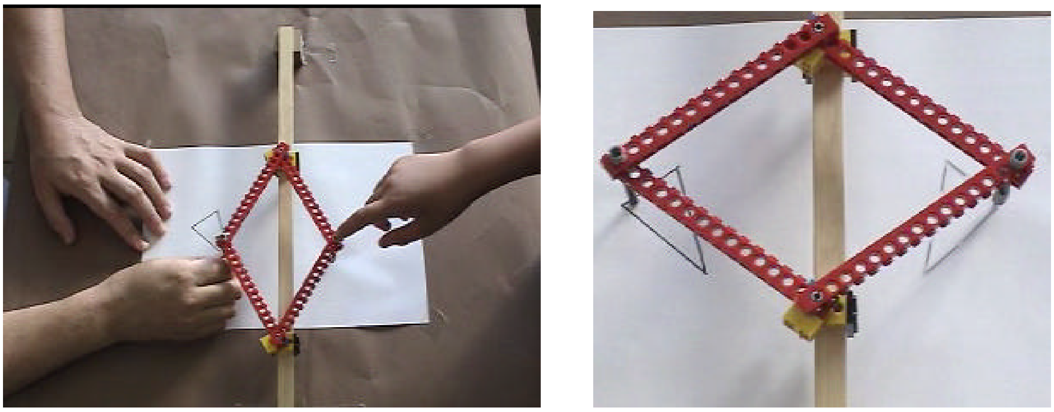

Figure 9. Line Symmetry

To help students design a mechanism, teacher asked students to explain the properties of line symmetry which they learned before and discussed the working images of mechanism (see figure 10). Students recognized mechanism should work same length from symmetry line and works perpendicularly with symmetry line. Then teacher gave each group a wire to design mechanism. Students in a few groups soon associated the structure with a pair of scissors because student's movement at figure 10 looked like using scissors. Other students could not imagine sooner and tried to imagine the mechanism by bending the wire (eg. figure 11). Then, many students imagined figure 12 inevitably. Finally, students design the structures (figure 13). In figure 13, right one is designed from the image of scissors and left one is designed from the image of rhombus that is already investigated at the Sylbester's pantograph at the last class. Students explained the reason why it must work as a drawing machine as line symmetry with the metaphor of scissors and both design includings the structure of rhombus. Students explained scissors and rhombus are functioning perpendicularly with symmetry line.

Figure 10

Figure 11.

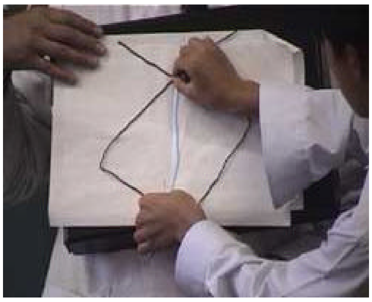

Figure 12

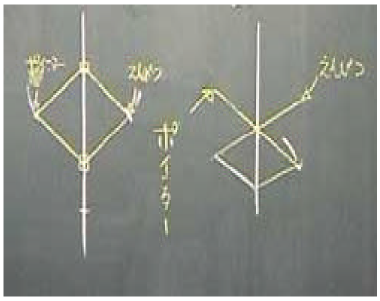

Figure 13

After students gained some confidence about the structure of mechanics from figure 13, teacher gave students LEGO model of figure 12. Via drawing the symmetry figures (figure 14), they confirmed their assumption of design was an appropriate assumption. At this stage, students felt LEGO mechanics was a proof because they had not yet learned formal proof. LEGO mechanics functioned as a demonstration tool. Some of the students tried to change parameter such as shown in figure 14. It stretches figure.

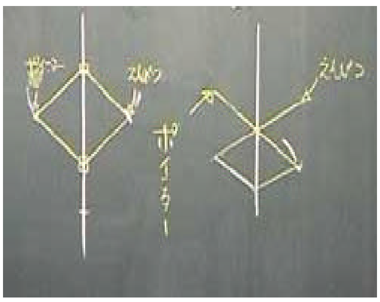

Figure 14

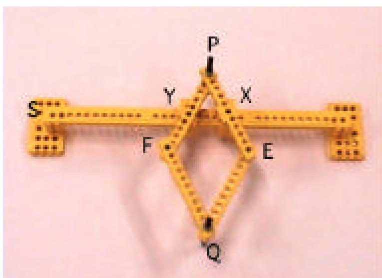

Figure 15

This example demonstrates that in the exploration LEGO mechanism, intermental functions mediated by LEGO mechanics also transferred and worked intramental functions in each student's mind.

Final Remark

Many mathematics teacher believe that mathematical representations are algebraic representations. One of the remarkable points in this paper is that LEGO mechanics also function as representation for mathematics such as object, subject and method for mathematical inquiry. This perspective is also supported by both of historical evidences and today's technology such as multiple representation mathematical software. LEGO mechanics as representation for mathematics has a peculiar intuition which enables students 'analysis by mechanics' and it internalizes the use of mechanics for mathematics. On the other hands, LEGO mechanics functions as physical representation such as moment and dead point. LEGO mechanics is not specific mathematical representation but functioning as mediational means for mathematics.

Acknowledgment

Special thanks to Jere Confrey and Maria G. Bartolini Bussi. The LEGO project could not start without their research projects.

Notes

1) This project report is rewrote from the earlier version of the paper that was presented at the Topic Study Group 5 at ICME9, 2000.

2) Project Manager, Institute of Education, University of Tsukuba.

3) The Junior High School Attached to the University of Tsukuba.

4) The Senoir High School Attached to the University of Tsukuba.

5) Faculty of Education, Shizuoka University.

6) Master Program of Education, University of Tsukuba.

7) Doctor Program of Education, University of Tsukuba.

8) LEGO Project is supported by the grant of JSPS. The grant number is 10558032.

References

Bartolini Bussi M., Nasi D., Martinez A., Pergola M. Zanoli C. & al. (1999). Laboratorio di Matematica: Theatrum Machinarum, I CD rom del Museo (1), Modena: Museo Universitario di Storia Naturale e della Strumenntazione Scientifica.

Bartolini Bussi M. (1996). History in the Mathematics Classroom. In Janke H. (Ed.), History of mathematics and education, Vandenhoeck & Ruprecht.

David Dennis & Jere Confrey (1995). Functions of a Curve: Leibniz's Original Notion of Functions and its Meaning for the Parabola, The College Mathematics Journal, **26**(**3**), 124-131.

David Dennis & Jere Confrey (1997). Drawing Logarithmic Curves with Geometer's Sketchpad: A Method Inspired by Historical Sources. In King J. Schattshneider (Ed.), Geometry Turned on (pp. 147-156), The Mathematical Association of America.

Isoda M. (Ed.). (1993-2000). Annual Reports of Algebra, Geometry and Calculus for All (Vols. 1-7), Tsukuba University. (written in Japanese)

Isoda M. (1997). Connecting Mathematics with Machine Engineering and Art: Perspectives for Calculus and Geometry for All via Technology. In Yang W. & Hasan Y. (Eds.), Proceedings of the Second Asian Technology Conference in Mathematics (pp. 60-70), Universiti Sains Malaysia.

Isoda M. (1998a). Mathematical Inquiry Enhanced by Harmonized Approach via Technology. In Park H., Choe Y., Shin H. & Kim S. (Eds.), Proceedings of ICME-EARCOME 1 (Vol. 3, 267-278).

Isoda M. (1998b). Developing the Curriculum for Curves Using History and Technology. In Yang W., Shirayanagi K, Chu S. & G. Fitz-Gerald (Eds.), Proceedings of the Third Asian Technology Conference in Mathematics (pp. 82-89), Springer.

Isoda M., & Matsuzaki A (1999). Mathematical Modeling in the Inquiry of Linkages Using LEGO and Graphic Calculator: Does New Technology Alternate Old Technology? Proceedings of the Forth Asian Technology Conference in Mathematics (pp. 113-122), ATCM Inc.

Isoda M., & Matsuzaki A. (2000). How can students mathematize the mechanics of motion with technological means?, Paper presented at WGA11, ICME9.

Kempe A. (1867). How to draw a straight line, Macmillan.

Otte M. (1991). Towards a Social Theory of Mathematical Knowledge. In Keitel C. & Ruthven K. (Eds.), Learning from Computers: Mathematics Education and Technology, Springer-Verlag.

Maanen van J. (1991). HOpital's Weight Problem, For the Learning of Mathematics, **11**(2), 44-47.

Maanen van J. (1992). Seventeenth Century Instruments for Drawing Conic Sections, The Mathematical Gazette, **76** (476), 222-230.

Smith D. (1967). A Source Book in Mathematics, Dover edition.

Wertsch J. (1991). Voices of the Mind, Harvard University Press.

URL

http://130.158.186.11/mathedu/forAll/index.html

http://155.185.1.61:80/labmat/usa1.htm

http://www.museo.unimo.it/theatrum/