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STRATEGIES FOR ADDITION AND SUBTRACTION OF WHOLE NUMBERS EXTENDED TO NUMBER SENTENCES INVOLVING FRACTIONS AND DECIMALS

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In this lesson study, Year 6 students (age 13 years) are asked to simplify, "using any appropriate strategy", eight addition and subtraction problems. The teacher also asks students "to communicate your thinking in your working".

WHAT DOES THIS LESSON SHOW?

1. When asked to simplify and solve sentences of the form $a + b$ and $a - b$, students have to think deeply about the meaning of "sum" (or total) in the case of $a + b$, and about "difference" in the case of $a - b$. (The expression "sum" is frequently used by teachers and in curriculum documents in describing sentences of the form $a + b$. The expression "difference" is less common. "Difference" is an important word because it expresses a relation between two numbers.)

2. This teacher uses the expression "*difference*" or "*find the difference between two numbers*" when asking students to consider sentences of the form $a - b$. Many other teachers would read a sentence of this form as "taking b away from *a*", or as "subtracting *b* from *a*", or even as "*a* minus *b*". While these are correct expressions, they express an operation, not a relation. The teacher's careful use of the relational word "*difference*" will be shown in the lesson.

3. The lesson is important because it builds upon ideas of relational thinking that the teacher has taught this year, and last year to those students who were in his Year 5 class. The nature of relational thinking, in contrast to computational or algorithmic-based thinking, will be shown in this lesson.

4. The lesson begins with number sentences involving only whole numbers, but in the second half of the lesson students have to think more deeply about addition and subtraction involving fractions and decimals.

5. Students apply strategies they have used to simplify addition and subtraction sentences with whole numbers to sentences involving fractions and decimals. Here, students have to think deeply about the nature of decimal and fraction numbers.

6. In discussion surrounding the eight questions, students are asked to communicate their reasons for choosing a particular strategy, and are asked to consider if there may be alternative strategies. Students are asked to comment upon strategies used by other students even though they may not have thought of using these strategies themselves.

7. Most of the strategies discussed by the teacher with the students are intended to train students to see that the numbers in sentences such as $a + b$ and $a - b$ are open to dimensions of variation (Marton & Morris, 2002; Marton, Runesson & Tsui, 2004),

and that recognising and using these degrees of variation is an important tool in simplifying and solving number sentences.

8. Helping students to see dimensions of variation in addition and subtraction sentences involving whole numbers, decimals and fractions builds up algebraic thinking. Seeing that numbers within number sentences can vary is intended to provide a bridge between the study of arithmetic and the emerging study of algebra.

9. The teacher's focus is having students see possibilities for simplification and successful calculation, and being able to explain why the simplification is appropriate. The teacher recognizes that some students will prefer to use strategies (i.e. algorithms that these students have learned in Years 2, 3 and 4) that do not rely upon relational thinking. All students are encouraged to consider other possibilities.

COMMENTARY ON THE LESSON RELATING TO EACH PROBLEM

Prior to the lesson, students were given an assignment entitled: Addition and subtraction strategies. They were asked to "Simplify using any appropriate strategy. Communicate your thinking in your working" for the following problems.

Problem 1: 826 – 489

The teacher commences by asking students, "What should you be thinking about before you start trying to work it out?" Some students say that they need to simplify the problem. The teacher asks again, "What are we trying to find?" Several students refer to a "difference". The teacher emphasises this point: "Any time we have a subtraction problem we are finding the *difference* between the numbers", then asks: "How have you thought it out?" A student says that he added 11 on to both numbers:

826 – 489
+11   +11

Before proceeding with the calculation, the teacher again asks, "Why did you add 11 to both numbers?" A student replies, "If you add eleven to the first number it makes the first number eleven more, so you have to add eleven to the second number". The teacher helps to finish the sentence by saying, "In order to …". Another student completes the sentence, "It keeps the *difference* the same".

The teacher asks, "Is there an alternative to adding eleven to each number?" A student says that it is possible to subtract 26 from both numbers, but agrees with the teacher that this would not result in making the problem easier.

The problem is completed, using an equal sign under the above working:

= 837 – 500
= 337

The teacher concludes by saying, "I know there are a number of you that are not using this strategy yet, but you can see [what it means]."

Problem 2: 9124 + 6968

Again the teacher asks students to think about what this problem is asking them to do. One student replies that it is asking them to find 6968 more than 9124. The teacher says, “We are finding the sum”.

Students described how they simplified the problem: “Add 32 to the second number”. The teacher explains, “Adding 32 to the second number makes the *sum* 32 more than it should be, so we have to take 32 from the 9124, giving 9092”.

9214 + 6968
– 32   +32
= 9092 + 7000
= 16092

One student explains that it is possible to add 1000 first to the 9192 and then add 6000, giving the same result. The teacher accepts this possibility. He then concludes, “The reason why we are using these strategies is that they make it easier”.

Problem 3: 3004 – 1746

The teacher asks first, “Can we use the same strategy here as we did in the first case?” Students agree that the goal is to make the second number into “a nice round number”. This gives rise to the suggestion that they could add 54 to both numbers (resulting in the second number becoming 1800). The teacher then asks, “Why do we need to add 54 (to the first number) and not take away?” Students agree that the *difference* has to be kept the same, giving:

3004 – 1746
+54   +54
= 3058 – 1800
= 1258

One of the students suggests that 3058 – 1800 could be simplified further by adding 200 to both numbers:

3058 – 1800
+200   +200
= 3258 – 2000
= 1258

The teacher accepts this suggestion, and goes on to ask, “Is there another way we can approach this problem? … Generally, we look at the *second* number to make it easier.” He is hinting that students think about transforming the first number. A student suggests taking 5 from each number, giving

3004 – 1746
– 5   – 5
= 2999 – 1741
= 1258

The teacher asks them to think why *nine* is the easiest number to subtract from. He says that some students might set out 2999 – 1741 formally (in vertical form). Students and teacher go through the formal steps: "1 unit from 9 units; 4 tens from 9 tens; 7 hundreds from 9 hundreds; and 1 thousand from 2 thousand".

Problem 4: 4024 + 7659

The teacher commences: "We are back to addition. We are finding a *sum* or *total*. What might we do here?" Students suggest subtracting 24 from the first number:

4024 + 7659
– 24   +24
= 4000 + 7683
= 11683

Asking students to compare this strategy (for addition problems) to those used with subtraction or difference problems, the teacher asks, "With addition, do we need to focus on any one of these numbers? No. We have got the choice. You can focus on either one of them. As long as you make one a nice round number to work with". (This contrasts with the subtraction problems where it was agreed "generally to make the second number a nice round number".)

The teacher reminds students that the equals sign can be used only at the beginning of equivalent lines. This last comment is important for understanding a student who said that he added 1 to 7659 and subtracted 1 from 4024, and then added 4023 + 7660 "bit by bit". The teacher asked, "You didn't use the equals sign (to connect lines) did you?" The student said he didn't. He said that he worked out the thousands first and then calculated the other parts of the sum "bit by bit":

4000 + 7000 = 11000
11000 + 600 = 11600
11600 + 60 + 23 = 11683 (Note that equals signs is used only to show results.)

Problem 5: $7\frac{1}{6} - 3\frac{5}{6}$

The teacher and students note that they are finding the difference between the numbers. The teacher asks, "Which number are we going to focus on - to make it a nice round number? Not that you can't use the first number. But it's generally easier the other way". This leads to agreement to add one-sixth to both numbers, giving

$7\frac{2}{6} - 4$
$= 3\frac{1}{3}$

One student says that he did it another way. He said that he converted seven and one sixth to six and seven sixths. The teacher asks him to explain. He says that he converted one of the "wholes" in 7 into six sixths, giving

$6\,\frac{7}{6} - 3\frac{5}{6}$

$= 3\,\frac{2}{6}$

This student understands that he is taking away 3 wholes and also taking away 5 sixths. After hearing this second approach, the teacher adds, "You have so many alternatives. You don't need to do it one particular way."

Problem 6: $12 - 7\frac{4}{9}$

Here the teacher does not remind students that they are dealing with a difference. He asks what number could be added to both numbers in order to simplify the problem. Students see that adding five ninths is the best way to simplify the problem:

$12 - 7\frac{4}{9}$

$= 12\,\frac{5}{9} - 8$

$= 4\,\frac{5}{9}$

The teacher then says: "Some of you could work this (problem) out in your heads. But what goes down on paper communicates how you thought it out". Taking a lead from this comment, a student explains his approach as follows:

$12 - 7 - \frac{4}{9}$

$= 5 - \frac{4}{9}$

$= 4\,\frac{5}{9}$

Teacher adds, "Fine. You have realised that you have to subtract the seven *and* the four ninths".

Problem 7: 8.23 – 3.67

The teacher first asks students to explain the meaning of each of the decimal numbers. "What is .23?" Students: twenty three hundredths. "What is the 2 on its own?" Students: two tenths. "What is the 3?" Students: three hundredths. Then the teacher continues: "We are finding the difference again. What is the easiest number to focus on?" He adds quietly, "That is an interesting question", knowing that some students will take this as a hint that they might possibly focus on the first number.

Students agree that making the second number into a "nice round number" would be a useful simplification to start with:

8.23 – 3.67
+.33   +.33
= 8.56 – 4.00
= 4.56

Taking the teacher's earlier hint, one student suggests taking .23 from both numbers:

8.23 – 3.67
–.23   – .23
8.00 – 3.44

The teacher asks, "Is this going to be easier?" No one agrees. The teacher asks, "Could we make it even easier?", hinting that the above simplification has not gone far enough. Several students suggest subtracting .24 from each number:

8.23 – 3.67
–.24   – .24
7.99 – 3.43

"We are subtracting from *nines* again", says the teacher. Together the class repeats:

"3 hundredths from 9 hundredths, (giving) 6 hundredths."

"4 tenths from 9 tenths, (giving) 5 tenths."

"3 wholes from 7 wholes, (giving) 4 wholes."

Problem 8: 7.06 + 9.892

Reminding the class that this is addition, the teacher asks, "What number are you going to focus on?" Someone suggests adding .008 to 9.892 (to make it 9.900):

7.06 + 9.892
–.008   + .008

Without proceeding any further, the teacher asks, "Who will find this easier? Maybe, it's not such an easy strategy to use." The teacher adds, "If you start thinking to do it one way, you don't have to keep doing it that way". Some students see this as a hint to look at the first number. Some suggest subtracting .06 from the first number. The teacher asks them: "If you take .06 from this number, what do you have to do to the other number?" They reply that it is necessary to add .06 to the second number:

7.06 + 9.892
–.06   + .06

The teacher and students work together as they "add 6 hundredths to 9 hundredths", giving 15 hundredths. "From 15 hundredths, we can make 1 tenth and 5 hundredths".

7.06 + 9.892
–.06   + .06

= 7 + 9.952
= 16.952.

The teacher concludes the lesson: “[There are] many strategies to work. Textbook (formal) strategies are included. Sometimes it might be easier to use them. [You need to ask] ‘What is going to be the best way of doing any particular problem?’ Now you have many strategies for doing that.”

HOW TYPICAL ARE THESE TEACHING APPROACHES IN AUSTRALIA?

In answering these questions, several points need to be made:

1. The school in which this lesson study has been captured is a private school and the strategies used by the teacher are not typical of many other teachers in the school. These teachers use computational approaches and, while encouraged to do so, have not introduced relational thinking into their mathematics classes.

2. The teacher in this study also gives attention to computational and algorithmic strategies for addition and subtraction. He is one of a growing group of elementary and junior secondary teachers who are moving arithmetic away from an almost exclusive focus on computational algorithms in order to foster students’ algebraic thinking. In this way, teachers are making a more effective transition between number patterns and relationships in arithmetic and some key ideas of algebraic thinking which students in the upper elementary school are expected to meet.

3. Australian national and state curriculum documents all recognize the importance of teaching students to use reliable written methods for computation. All curriculum documents also emphasize that mental computation has an important place along side the teaching of algorithms. There is no clear agreement among the various national and state documents about when formal written algorithms for the addition and subtraction of multi-digit numbers should be introduced into the elementary school mathematics program. Some states appear to favour a later introduction than others.

4. Authors such as McIntosh (in press) argue that mental computation should be given greater priority in the elementary school curriculum, stating that “when children calculate mentally, they use conceptual understanding of the numbers and operations involved, unlike the use of formal algorithms, which draws on memory of rules” (p. 4). McIntosh even argues “in favour of placing mental computation, instead of written computation, at the heart of primary school computational work” (p. 3). This position is more radical than that recommended in any of the national and state curriculum documents. McIntosh is quite careful to point out that his position is quite different from “advocating an even heavier diet of mental computation and tests which have formed the main approach to mental computation in the past” (p. 3).

5. McIntosh’s work has attracted the attention of national and state governments in Australia. In one national funded project led by McIntosh, the aim was to explore “in Grade 2, 3, and 4 classrooms ... ways of moving from mental computation to informal written computation, while refraining from teaching the formal written algorithms of

addition and subtraction, and about the effects on teachers and children and their advice to teachers in other schools as a result of this project.

6. National and state curriculum documents no longer advocate priority of place to the teaching of formal written algorithms for addition and subtraction, although these approaches are still recommended in most of the national and state documents.

WHAT IS SPECIAL ABOUT THIS LESSON STUDY?

While national and state documents all recommend the use of mental computation and the teaching of alternative written methods, there is no one recommended approach. All recommended approaches, however, deal only with addition and subtraction of whole numbers. This teacher encourages students to use relational strategies to simplify and solve addition and subtraction problems involving whole numbers, decimals and fractions.

Other teachers who use a similar approach to addition and subtraction have found it especially helpful to some students who are experiencing difficulty in carrying out formal written algorithms. This is a very important point. While some students being taught in this lesson study may appear to be quite able, it should be remembered that the whole class is a mixed ability class, with some students who find mathematics difficult. Other teachers report that the strategies used in this lesson are accessible and successful with many students in Years 4 to 8 who are not mathematically able.

What does this approach have to offer to teachers and students in other countries, especially those APEC countries who are striving to build up the proportion of students entering and successfully completing secondary education?

These reforms require a rethinking of the elementary mathematics curriculum. In the past, because participation in secondary education was very limited, the elementary school mathematics curriculum focused almost exclusively on computation proficiency and the teaching of algorithms. This approach is unlikely to provide the growing proportion of young people entering secondary school with the mathematical experiences needed to understand algebraic structure and reasoning in the secondary curriculum. If more young people are to succeed in secondary school, then they must leave elementary school with a deeper experience of relational (algebraic) thinking which can be developed through number sentences and operations.

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